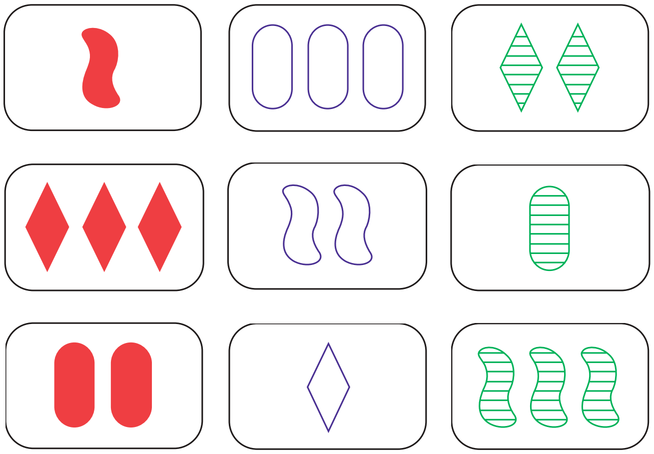
MAGIC SQUARES

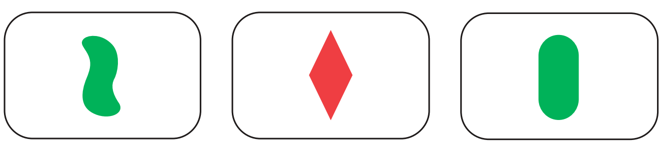


What you see here is a magic square, much like the addition and subtraction squares you may have used as a child.

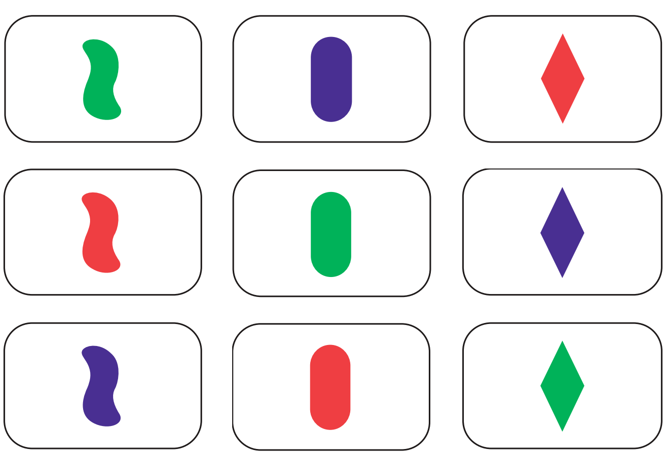
These magic squares are even more talented, as they all follow the rules of the card game SET®. To learn how to make one with ease, read on.

SET® cards contain four properties: **color**, **shape**, **number** of objects, and **shading**. The rules state for each property, they must all be equal, or all different. For example, if we look at the top row of the square, we see three different colors, three different shapes, three different numbers, and three different types of shading within the objects. Need more examples? Any line on the magic square yields a set. Constructing a magic square may seem complex at first glance, but in reality anyone can make one by following this simple process:

- *Choose any three cards that are not a set. (It will work with a set but the square becomes redundant) For example, we will choose these:*



- *Using our powers of deduction, we can conclude that in order to create a set in the first row, the #2 card needs to have a different color, different shape, same number, and same shading as the #1 and #3 cards. That leaves us with a solid purple oval. The rest of the square can be completed in the same way, giving us the following magic square:*



A few examples will convince you that this method works. Not only does the magic square work but it can be theoretically proven through a mathematical model. This model makes an easy proof of the magic square as well as answer any questions about how SET® works.

MATHEMATICAL PROOF OF THE MAGIC SQUARES

By Llewellyn Falco

One day, while sitting by myself with a deck of SET® cards, I began to wonder whether or not I could construct a 3x3 square which made a set regardless of which direction you looked. I sorted the deck into single colors, and then started constructing a square. To my surprise it worked. I tried to make another one. It worked. As a test, I made a 3x3 square with all three colors, and sets involving no similarities, and other sets with only one difference. When it ended up working out I was convinced that no matter which cards you started with, you could always construct a 3x3 square that made a set in every direction. Being an educated man, and a lover of mathematics, I decided that I should be able to prove this theory. So I set out to work; this is the fruit of my labor... First, we need a convention in which to label the cards. Thus, if we look at each characteristic on each card separately, and denote all variation to 1, 2, or 3...

Number[X_1] Color[X_2] Symbol[X_3] Shading[X_4] $\epsilon\{1,2,3\}$

So the vector x=[p,q,r,s] completely describes the card.

For example: the card with one, red, empty, oval might be Number[X_1] = 1, Color[X_2] =1, Symbol[X_3] = 1, Shading[X_4] = 1, or x = [1,1,1,1].

For shorthand, I use the notation C_X to represent the card.
Where C_X = Number[X_1] , Color[X_2] , Symbol[X_3], Shading[X_4], and x = [p,q,r,s].

If I wanted to make the third card which makes a set from two cards Ca and Cb, I would have the card $C_{(ab)}$ where
ab = [a_1b_1, a_2b_2, a_3b_3, a_4b_4]

and the rule for the operator is:
If an = bn, then bn = xn and an = xn If an≠bn, then bn≠xn and an≠xn

For Example: 1*1=1, 1*2=3, 1*3=2
[1,1,2,2] [1,2,2,3] = [1,3,2,1]

This holds consistent with the rules of SET®. If the first two cards are red the third must also be red; if the first one is a squiggle, and the second a diamond, the third must be an oval.

Here are some basic theorems in this group and their proofs:

<table>
<tr><td rowspan="2">$a_n * b_n = b_n * a_n$</td><td colspan="2">Proof 1.1
Two cases: $a_n = b_n$, $a_n \neq b_n$</td></tr>
<tr><td>Case 1:
$a_n = b_n$
1*1 = 1*1</td><td>Case 2:
$a_n \neq b_n$
1*2 = 2*1</td></tr>
</table>

<table>
<tr><td rowspan="2"></td><td>1 = 1</td><td>3 = 3</td></tr>
<tr><td colspan="2">Note: this just shows that any two cards make a third, regardless of order.</td></tr>
<tr><td>$(a_n * b_n)\, c_n \neq a_n * (b_n * c_n)$</td><td colspan="2">Proof 1.2
$(a_n * b_n)\, c_n \neq a_n * (b_n * c_n)$
$3(2 * 1) = (3 * 2) * 1$
$3 * 3 = 1 * 1$
$3 \neq 1$</td></tr>
<tr><td rowspan="5">$(a_n * c_n)\,(a_n * b_n) = a_n(c_n * b_n)$</td><td colspan="2">Proof 1.3
There exist four options:
$a = b = c$
$a = b,\ b \neq c$
$a \neq b,\ b = c$
$a \neq b \neq c$</td></tr>
<tr><td>$a = b = c$</td><td>$(1*1)(1*1) = 1*(1*1)$
$(1*1) = (1*1)$
$1 = 1$</td></tr>
<tr><td>$a = b,\ b \neq c$</td><td>$(1*1)(1*2) = 1*(1*2)$
$(1*3) = (1*3)$
$2 = 2$</td></tr>
<tr><td>$a \neq b,\ b = c$</td><td>$(1*2)(1*2) = 1*(2*2)$
$3*3 = 1*2$
$3 = 3$</td></tr>
<tr><td>$a \neq b,\ b \neq c$</td><td>$(1*2)(1*3) = 1*(2*3)$
$3*2 = 1*1$
$1 = 1$</td></tr>
<tr><td rowspan="3">$a(a * b) = b$</td><td colspan="2">Proof 1.4
Two cases exist: $a = b$ or $a \neq b$</td></tr>
<tr><td>Case 1:
If $a = b$
then $1(1*1) = 1$
$1*1 = 1$
$1 = 1$</td><td>Case 2:
If $a \neq b$
Then $1(1*2) = 2$
$1*3 = 2$
$2 = 2$</td></tr>
<tr><td colspan="2">Note: This is just a case that a set of three cards can be made by taking any two of the three cards a*b=c means b=c*a means a=b*c, now we see that (a*b) =a*b, then move to the other side…a(a*b)=b</td></tr>
</table>

The Square

So let us begin by choosing any three cards: a, b, and c, and placing them in positions 7, 5, 9.

1	2	3
4	**C_c**	6
C_a	8	**C_b**

Now we need to fill in the blanks for the remaining cards. Starting with card 8; it needs to complete the set with the cards **C**a and **C**b. We now look at the multiplier. The new card will be the product of **C**a operating on **C**b which is **C**ab. Likewise, filling in slots 1 and 3 leaves us with the square below.

C_{bc}	2	**C_{ac}**
4	C_c	6
C_a	**C_{ab}**	C_b

We now know that we have three sets on the board (7,8,9; 7,5,3; 1,5,9). However, how should we go about filling slot two? I chose to combine 5 and 8 and give slot 2 the card **C**c(ab). Now I have a set going down 2,5,8, but the theory states that 1,2,3 should form a set as well. This means if I choose to combine **C**bc with **C**ac, it would equal **C**(bc)(ac), which must be the same card as **C**c(ab). Therefore we must show that:

(bc)(c(bc)) = (ac)

c(b(ab)) = ac	by 1.3 and 1.1
c(a) = ac	by 1.4 and 1.1
ac = ac	by 1.1

Now we fill in the two remaining slots of 4 and 6 by combining down to end up with 4 equaling **C**a(bc) and 6 equaling **C**b(ac). So now we have the following square:

C_{bc}	**$C_{c(ab)}$**	C_{ac}
$C_{a(bc)}$	C_c	**$C_{b(ac)}$**
C_a	C_{ab}	C_b

Now that all other rows, columns, and diagonals have been accounted for, we only have to prove that 4,5,6 is a set. This means

(a(bc))*c = b(ac)

b(bc) = c	by 1.4
(a(bc))(b(bc)) = b(ac)	substitution of b(bc) for c
(bc)(ab) = b(ac)	by 1.3

(bc)(ba) = b(ac)   by 1.1
b(ca) = b(ac)   by 1.3
b(ac) = b(ac)   by 1.1

This completes the proof of the square. Four sets still remain unaccounted for (1,6,8; 3,4,8; 7,2,6; 9,2,4). We note that if we prove one of these all must be true since now we can reconstruct this square by placing any card from 1,3,7, or 9 in the beginning slot and still get the same square.

Proof of the *SET* 1,6,8

C_{bc}	**$C_{c(ab)}$**	C_{ac}	(bc)(ab) = b(ac)(bc)
$C_{a(bc)}$	C_{c}	**$C_{b(ac)}$**	(ba) = b(ac)
C_{a}	C_{ab}	C_{b}	b(ac) = b(ac)

Proof of the *SET* 3,4,8:

C_{bc}	$C_{c(ab)}$	**C_{ac}**	(ac(a(bc)) = ab
$C_{a(bc)}$	C_{c}	$C_{b(ac)}$	a(c(bc)) = ab
C_{a}	**C_{ab}**	C_{b}	a(b) = ab

Proof of the *SET* 7,2,6:

C_{bc}	**$C_{c(ab)}$**	C_{ac}	a(c(ab)) = b(ac)
$C_{a(bc)}$	C_{c}	**$C_{b(ac)}$**	(ab))(c(ab)) = b(ac)
C_{a}	C_{ab}	C_{b}	(ab)(bc) = b(ac)
			b(ac) = b(ac)

Proof of the *SET* 9,2,4:

C_{bc}	**$C_{c(ab)}$**	C_{ac}	(b(c(ab)) = a(bc)
$C_{a(bc)}$	C_{c}	$C_{b(ac)}$	(a)(ab))(c(ab) = a(bc)
C_{a}	C_{ab}	**C_{b}**	(ac)(ab) = a(bc)
			a(bc) = a(bc)

The largest group of cards you can put together without creating a set is 20. By following this method, you'll understand how.

- *If we choose just two characteristics of a card (for example, shape and number), we can then plot it on a matrix.*

	1	2	3
Squiggle	•		
Diamond			
Oval			

- *So now we can see if three cards form a set by noticing if they make a line on the matrix, as shown below. These three cards all have one object and different shapes.*

	1	2	3
Squiggle	•		
Diamond	•		
Oval	•		

- *Similarly, the following lines all produce sets, even if they wrap around the matrix in space like the one on the far right.*

	1	2	3
Squiggle			•
Diamond		•	
Oval	•		

	1	2	3
Squiggle			
Diamond			
Oval	•	•	•

	1	2	3
Squiggle		•	
Diamond			•
Oval	•		

- *Now the following matrix shows us how many squares we can fill without creating a line:* **4**.

	1	2	3
Squiggle	•		•
Diamond			
Oval	•		•

- *Now we can add a third characteristic, color...and we can think of the matrix as a depiction of a 3D tic tac toe board. You can see below how to plot 9 cards without a set.*

	1	2	3
Squiggle	●		●
Diamond			
Oval	●		●

Red

	1	2	3
Squiggle		●	
Diamond	●		●
Oval		●	

Green

	1	2	3
Squiggle			
Diamond		●	
Oval			

Purple

- *When using all four SET® card properties, we can plot a 20 card no set.*

	1	2	3
Squiggle	●		●
Diamond			
Oval	●		●

Red

	1	2	3
Squiggle		●	
Diamond	●		●
Oval		●	

Green

	1	2	3
Squiggle			
Diamond		●	
Oval			

Purple

solid

	1	2	3
Squiggle		●	
Diamond	●		●
Oval		●	

Red

	1	2	3
Squiggle	●		●
Diamond			
Oval	●		●

Green

	1	2	3
Squiggle			
Diamond		●	
Oval			

Purple

striped

	1	2	3
Squiggle			
Diamond		●	
Oval			

Red

	1	2	3
Squiggle			
Diamond		●	
Oval			

Green

	1	2	3
Squiggle			
Diamond			
Oval			

Purple

open