Investigations into the Card Game SET

THE MAXIMUM NUMBER OF SETS FOR N CARDS
AND
THE TOTAL NUMBER OF INTERNAL SETS FOR ALL PARTITIONS OF THE DECK

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Introduction

I never thought that a purchase of a card game would lead to any kind of mathematical adventure. However, it is evident, from the abundance of related articles and papers written, that SET has sparked an enormous amount of interest from non-mathematicians and mathematicians alike. Clearly, SET is no ordinary card game!

My first investigation into the mathematics behind SET began with a December 19, 1998 letter, written to the creators of the game, and concerning the odds statistics presented in the game's instructions. Being an actuary by profession and an individual with a passion for mathematics, I was immediately drawn to the question of how to calculate the probability of no Set (the special meaning of this will be defined later) being present in twelve cards. This problem naturally led to the more general question of how to determine the maximum number of cards that could be chosen without a Set existing. I soon recognized that this was an extremely complex problem for which computer assistance would be required to evaluate the seemingly endless possible variations. Famous math problems, like the Four Color Theorem, were solved by the use of the computer, so why not this one? After all, counting problems are well suited to computer modeling!

But I also continued to optimistically hold on to the notion that the problem of determining the largest number of cards without a Set, and the possible maximum number of Sets for a given number of cards, could be formulated in algebraic terms. However, because other individuals, included the creators of the game, had already provided solutions to the original SET problem, I decided that it probably wasn't worth any more investment of time.

Recently, my interest in the SET problem was rekindled after reading Mr. David Van Brink's highly inspiring article on the subject. In that article, he demonstrates that a deck of N = 47 cards must contain a Set. While reading this article, it became apparent that the concept of parity was fundamental to formulating an alternate approach that might uncover the mathematical patterns behind SET.

This paper will cover the following topics:

1. For the benefit of those new to SET, a brief explanation of the rules of the game and how Sets are formed.
2. Development of a general formula for the total number of Sets that can occur when a deck of C^P (P = number of properties in the deck, C = number of choices for each property) cards is partitioned into two piles, and Set counts are restricted to those that are found exclusively within each pile.
3. The thinking behind the clever visual solution for why any collection of 20 cards must contain a Set, including a discussion of the Set-blocking strategy.

4. A proposed computer modeling method, referred to as the Consecutive Maximization Method, for use in identifying the largest possible Setless collection of cards from a P property deck.

Rules of the Game SET

The card game SET is played with a deck of 81 cards, each bearing a picture that has four properties and three features for each property:

- Shape – diamond, oval, or squiggle
- Color – green, purple, or red
- Number – one, two, or three
- Shading – solid, striped, or open

By the rules of the game, twelve cards are laid out in a grid on the table and players try to simultaneously identify as many Sets as possible within those twelve cards. If a Set cannot be found, three more cards are added to the grid. A "Set" is defined as a group of three cards that, for each property, either has the same feature for all three cards, or a different feature for all three cards.

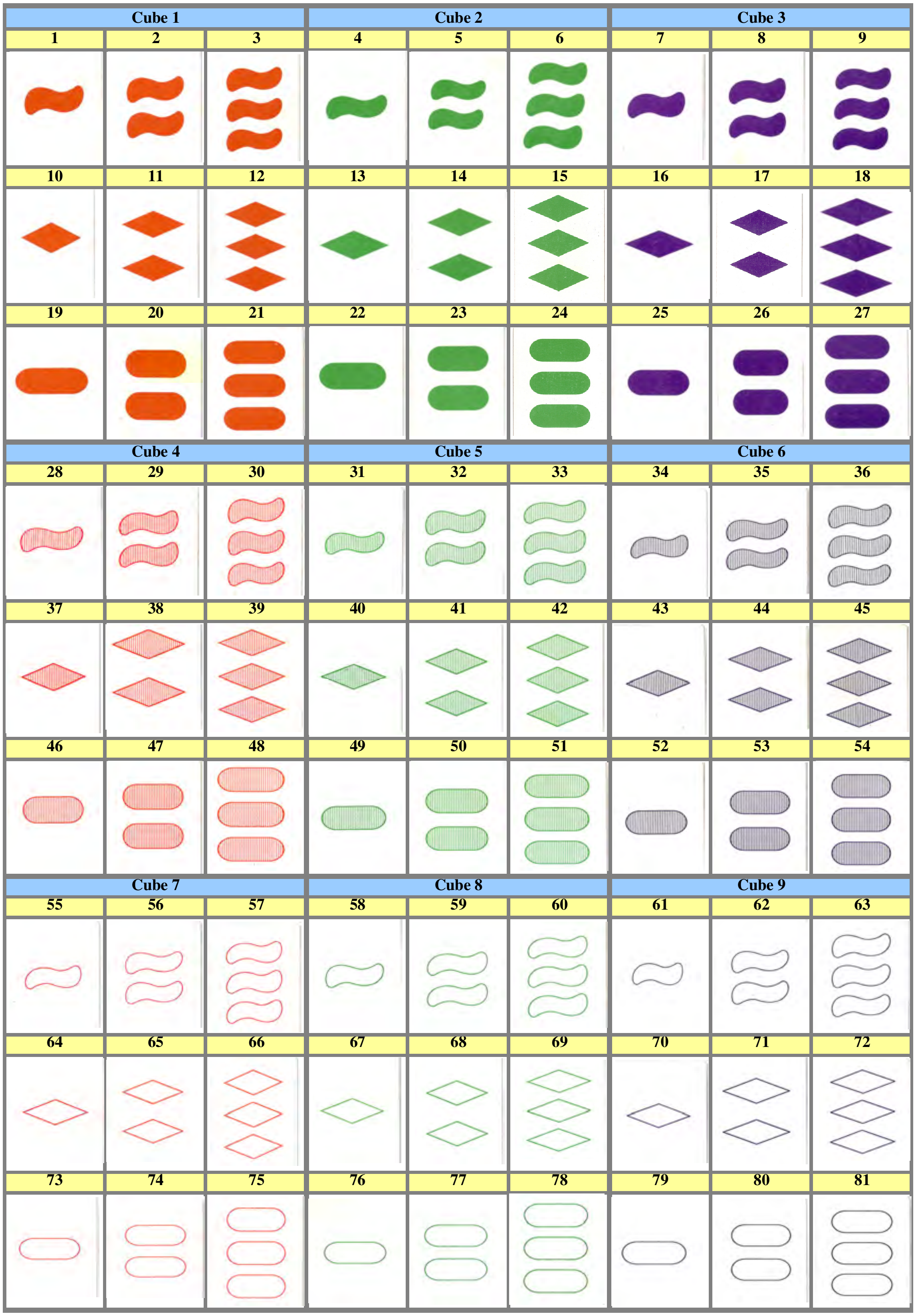
The illustration on the page 6 shows all 81 cards arranged by property in nine large "cubes", each containing 9 cards. Technically, the term "cube" is not accurate because it implies the presence of three dimensions, and what is pictured is a series of 3 x 3 squares. However, if we place card trays, each with nine compartments, over each 3 x 3 square, we can envision thin cubes! I will hereafter refer to each 3 x 3 arrangement of cards as a "cube" to emphasize its unique significance.

The cards are arranged so that color and number are represented in columns and shape and shading in rows. The table below provides a description of how the illustration was constructed.

Property	Feature	Representation
Color	Red	Columns 1, 2, and 3
Color	Green	Columns 4, 5, and 6
Color	Purple	Columns 7, 8, and 9
Number	One	Columns 1, 4, and 7
Number	Two	Columns 2, 5, and 8
Number	Three	Columns 3, 6, and 9
Shape	Squiggle	Rows 1, 4, and 7
Shape	Diamond	Rows 2, 5, and 8
Shape	Oval	Rows 3, 6, and 9
Shading	Filled in	Rows 1, 2, and 3
Shading	Striped	Rows 4, 5, and 6
Shading	Open	Rows 7, 8, and 9

Each feature for a particular property is represented by 27 cards, or one-third of the deck. It would be easy to expand the illustration to display additional properties (assuming there were still three features to choose from). For example, a fifth property could be what type of outline the shapes had – either solid, dashed, or no outline. This would require $3^5 = 243$ cards and a 27 x 9 array. A sixth property would require $3^6 = 729$ cards and a 27 x 27 array. Any deck of cards with P properties can be represented in a two-dimensional array using 3^{P-2} cubes, each containing 9 cards.

The SET[1] Deck of Cards



[1]SET is a trademark of SET Enterprises, Inc. The SET cards are depicted here with permission.

To get familiar with the game, consider the following examples of Sets that can be formed. The numbers in the following table refer to the identifying numbers that appear above each card in the illustration on the preceding page. If you own SET, it might make it easier to follow this discussion if you laid out the cards as indicated in the diagram (and if you don't own the game, well you know what to do!).

	Cards Forming a Set			
Property	**1-2-3**	**1-5-9**	**1-48-65**	**1-41-81**
Color	All Red	Red, Green, Purple	All Red	Red, Green, Purple
Number	One, Two, Three	One, Two, Three	One, Three, Two	One, Two, Three
Shape	All Squiggles	All Squiggles	Squiggle, Oval, Diamond	Squiggle, Diamond, Oval
Shading	All Filled In	All Filled In	Filled In, Striped, Open	Filled In, Striped, Open
Description of Set formed	1 property has different features, 3 properties have the same feature	2 properties have different features, 2 properties have the same feature	3 properties have the different features, 1 property has the same feature	All 4 properties have different features

It is instructive to note what happens after you have selected your first card. If you pick your second card from the same cube, the card that forms a Set with that pair of cards is within that same cube and "in line" with the first two cards. A "line" in this case includes 3 cards in a "wrap-around" formation in addition to the typical horizontal, vertical, or diagonal formations. For example, card 1 makes the following Sets within cube 1: 1-2-3 (horizontal), 1-11-21 (diagonal), 1-10-19 (vertical), and 1-12-20 (wrap around). Each of the 9 cards within a cube can make 4 Sets, for a total of 36 Sets per cube. However, each Set appears three times, so there are only 12 unique Sets per cube. Because there are nine cubes, there are 12 x 9 = 108 Intracube Sets (i.e., Sets that are formed by cards located in the same cube).

If you pick your second card from another cube, then the third card required to complete the Set will come from a different cube than either of the first two. For example, if you picked a card from cube 1 and another card from cube 5, the Set-forming card will not be found in either cube 1 or cube 5, or a situation would result in which only two of the three cards have the same property features. This is a violation of the rules of SET. In this particular instance, the card that completes the Set will be found in Cube 9.

Intercube Sets (i.e., Sets that are formed by cards in different cubes) have the same type of line formations as Intracube Sets, except that the lines are formed by the cube combinations. The 12 line-forming combinations of cubes that account for the remainder of the Sets are summarized in the following table.

Cube Combinations for Intercube Sets

Horizontal	Vertical	Diagonal	Wrap Around
Cubes 1, 2, and 3	Cubes 1, 4, and 7	Cubes 1, 5, and 9	Cubes 1, 6, and 8
Cubes 4, 5, and 6	Cubes 2, 5, and 8	Cubes 3, 5, and 7	Cubes 2, 4, and 9
Cubes 7, 8, and 9	Cubes 3, 6, and 9		Cubes 2, 6, and 7
			Cubes 3, 4, and 8

Each of the above cube combinations account for 9 x 9 = 81 unique Sets. The total number of unique Sets accounted for by these Intercube combinations is 81 x 12 = 972. Together with the 108 unique Intercube Sets, this makes a total of 1,080 unique Sets that can be formed by the 81 cards.

It is interesting to determine the number of different types of Sets that can occur in the game.

Case 1: All four properties have different features

Choose the first card: There are **81 choices**.

Choose the second card: Because there are only two choices left for each of the four properties, there are 2 x 2 x 2 x 2 = **16 ways to make this choice.**

The total number of unique Sets where all four properties have different features is

(81 x 16) ÷ 6 = 216

(Note that division by 6 is necessary because we do not care about the order of the three cards, and there are 3 x 2 = 6 ways to arrange three cards).

Case 2: Three properties have different features; one property has the same feature

Choose which three of the four properties have different features and which one property has the same feature: **4 ways to make this choice.**

For the property that has the same feature, make a choice from the three possible features for this property (e.g., if color is selected as the feature to be the same, you can select red, green, or purple): **3 ways to make this choice.**

Choose the first card: 3 x 3 x 3 = **27 choices because the feature for one property has already been chosen.**

Choose the second card: 2 x 2 x 2 = **8 choices because there is one less choice for each property.**

The number of unique Sets where three properties have different features and one property has the same feature is

(4 x 3 x 27 x 8) ÷ 6 = 432

Case 3: Two properties have different features; two properties have the same features

Choose which two of the four properties have different features and which two properties have the same features: **6 ways to make this choice.**

For each of the two properties which have the same features, make a choice from the three possibilities for that feature:

3 x 3 = **9 ways to make this choice.**

Choose the first card: 3 x 3 = **9 choices because features for two properties have already been chosen.**

Choose the second card: 2 x 2 = **4 choices because there is one less choice for each property.**

The number of unique Sets where two properties have different features and two properties have the same features is

(6 x 9 x 9 x 4) ÷ 6 = 324

Case 4: One property has different features; three properties have the same features

Choose which one of the four features are different and which three are the same: **4 ways to make this choice.**

For each of the three properties which have different features, make a choice from the three possibilities for that feature:

3 x 3 x 3 **= 27 ways to make this choice.**

Choose the first card: **3 choices because features for three properties have already been chosen.**

Choose the second card: **2 choices because there is one less choice for the property that is to differ.**

The number of unique Sets where one property has different features and three properties have the same features is

(4 x 27 x 3 x 2) ÷ 6 = 108

The table below summarizes the number of unique Sets for all four cases, along with suggested point values, that could be awarded during the game. These point values reflect the frequency with which the type of Set identified can occur.

Type of Set	**# of Unique Sets**	**Proposed Point Value for Game**
All four properties have different features	216	3
Three properties have different features; one property has the same feature	432	1
Two properties have different features; two properties have the same features	324	2
One property has different features; three properties have the same features	108	4
Total	**1,080**	

Terminology and Background for the Total Internal Set Formula

As a convention, I will use the middle dot as a multiplication symbol and a slash as a division symbol in all formulas.

Set – Like Mr. Van Brink, I will use the capitalized term "Set" to refer to the special three card collection, which is called a set by the rules of the game of SET.

Primary Pile – a collection of N cards that have been selected, where $3 \leq N < 81$. For discussion purposes, this is the pile of cards we desire to be Setless. Note that N is required to be strictly less than 81 because we are analyzing partitions of the deck, and N = 81 indicates no partition.

Complementary Pile – the 81−N card collection, which together with the Primary Pile, constitute the entire deck of 81 cards. For discussion purposes, this is the pile of cards used to block the formation of Sets within the Primary Pile.

Internal Set – a Set formed from all three cards, found entirely within a collection of N cards, where $3 \leq N \leq 81$.

Crossover Set – an external Set formed using two cards from a Primary Pile and one card from its Complementary Pile, or one card from a Primary Pile and two cards from its Complementary Pile.

We begin by noting that, by the rules of the game, a Set is defined by choosing a pair of cards because the third card which completes the Set is automatically determined by that choice. If you divide the deck of 81 cards into two piles, every Set is present either as an Internal Set or as a Crossover Set. If we randomly select a pair of cards from the Primary Pile, then either the third card to complete a Set is found in the Primary Pile or else it is found in the Complementary Pile. If it is found in the Primary Pile, then we have identified an Internal Set; otherwise, we have identified a Crossover Set. All possible states for the Sets created by a Primary Pile and its Complementary Pile are pictured below.

Type of Set	Primary Pile N Cards	Complementary Pile 81-N Cards
Internal		☐ ☐☐
Crossover	☐	☐ ☐
Crossover	☐ ☐	☐
Internal	☐ ☐☐	

We now derive key numbers, some of which are used as part of the total Internal Set formula.

Total number of Sets in the deck

A pair of cards A and B can be ordered two ways, AB or BA, but we do not care about card order in SET. If C is the third card that makes a Set with cards A and B, then ABC and BAC are identical Sets. There are a total of $(81 \cdot 80) / 2 = 3{,}240$ pairs of cards in the 81 card deck. Because pairs of cards also constitute a Set, there are also 3,240 total Sets in the deck.

Unique number of Sets in the deck

Because a pair of cards, A and B, determine a Set, and we can position the third Set member, say C, either before A, between A and B, or after B, it follows that every pair of cards, and, therefore, every Set, is counted three times in the total Set count. Therefore, there are 3,240 / 3 = 1,080 unique Sets in the 81 card deck.

Number of Sets per card

There are a total of 3,240 Sets in the deck of 81 cards, so every card is part of 3,240 / 81 = 40 Sets.

Set Counts for the Primary and Complementary Piles

Next, we will identify the unique number of Sets for each of the above diagrammed states. It is important to recognize that, when the deck is divided into two piles, the only Sets that get counted three times are the Internal Sets because, by definition, all three cards needed to create an Internal Set are found entirely within the Primary or Complementary Pile.

Counts for Total Number of Sets

The total number of Sets (Internal and Crossover) accounted for by either the Primary Pile or its Complementary Pile is the number of cards in the respective pile times 40 because there are 40 Sets per card. For the Primary Pile, the total number of Sets is $40 \cdot N$, and for the Complementary Pile, the total number of Sets is $40 \cdot (81\text{-}N)$. Every pair of cards within a Pile will be part of either an Internal or a Crossover Set.

Counts for Internal Sets

Assume that S_P represents the number of pairs forming Internal Sets for the Primary Pile, and S_C represents the number of pairs forming Internal Sets for the Complementary Pile. Then, the unique number of Internal Sets for the Primary Pile is S_P, and the unique number of Internal Sets for the Complementary Pile is S_C. The S_P pairs account for $3 \cdot S_P$ total Sets, and the S_C pairs account for $3 \cdot S_C$ total Sets because, as noted earlier, every pair is counted three times in the total Set count.

Counts for Crossover Sets

The number of unique Crossover Sets is equal to the number of Sets accounted for by all possible pairs, less the total number of Internal Sets. There are $[N \cdot (N-1) / 2] - 3 \cdot S_P$ unique Crossover Sets for the Primary Pile and $[(81-N) \cdot (80-N) / 2] - 3 \cdot S_C$ unique Crossover Sets for the Complementary Pile.

The Formula for the Total Number of Internal Sets for a Deck Partition

The following table summarizes the Set counts for the Primary and Complementary Piles. These counts are used to develop the total Set count formula.

Set Counts for Primary and Complementary Piles

Type of Set	Primary Pile N Cards	Number of Unique Sets	Complementary Pile 81-N Cards	Number of Unique Sets
Internal			□ □ □	S_C
Crossover	□		□ □	$[(81-N)\cdot(80-N) / 2] - 3\cdot S_C$
Crossover	□ □	$[N\cdot(N-1) / 2] - 3\cdot S_P$	□	
Internal	□ □ □	S_P		

It is important to recognize that there will always be 1,080 unique Sets present between the two piles. Sets never disappear from the deck – they just appear in the form of a Crossover! However, our primary interest is in Internal Sets – the ones that can occur exclusively within a pile of cards.

Totaling the formulas highlighted in the table yields the following equation:

$[N\cdot(N-1) / 2] - 3\cdot S_P + S_P + [(81-N)\cdot(80-N) / 2] - 3\cdot S_C + S_C = 1,080$

$2\cdot(S_P + S_C) = [N\cdot(N-1) / 2] + [(81-N)\cdot(80-N) / 2] - 1,080$

$2\cdot(S_P + S_C) = (N^2 - N + 6,480 - 161\cdot N + N^2) / 2 - 1,080$

$2\cdot(S + S_C) = (4,320 + 2\cdot N^2 - 162\cdot N) / 2$

$S_P + S_C = (N^2 - 81\cdot N + 2,160) / 2$

The last formula indicates that the total number of unique Internal Sets for the Primary Pile and its Complementary Pile is $(N^2-81\cdot N+2,160) / 2$. A graph of this function is provided on page 14. This function is useful because, it tells us that, if a Primary Pile of N cards does not contain any Internal Sets, its Complementary Pile must contain exactly $(N^2-81\cdot N+2,160) / 2$ Internal Sets.

It is also worth noting that the above formula could be derived directly by realizing that, before we begin partitioning the deck, there are 1,080 unique Sets within the 81 cards. For every card that is removed from the original deck and transferred to a second pile, we initially subtract 40 Sets from the Set count because each card accounts for 40 Sets. However, we must add back an adjustment equal to the number of pairs formed from the N cards placed in the second pile because we gain either Crossover or Internal Sets, from the transfer, equal to the number of pairs occurring in this pile. In mathematical terms, we have

$$S_P + S_C = 1,080 - \underline{40\cdot N + N\cdot(N-1) / 2} = (N^2 - 81\cdot N + 2,160) / 2$$

This formula is identical to the one above! One final observation is that the underlined terms in the above equation can be combined in the following way:

$$N\cdot(N-1) / 2 - 40\cdot N = (N^2 - 81\cdot N) / 2 = N\cdot(81-N) / 2$$

The last expression represents the total number of Crossover Sets. In words, the formula we have developed states:

Total number of Internal Sets = Total number of unique Sets – Total number of Crossover Sets
or
Total number of unique Sets = Total number of Internal Sets + Total number of Crossover Sets

General Formula for the Total Internal Set Count (Number of Properties = P, Number of Choices = C)

The reasoning used in the development of the total internal SET formula for a deck of cards with 4 properties and 3 choices per property can be extended to produce a general formula for a C^P card deck, where there are P properties and C choices per property, but the rules for forming Sets are the same as those for the SET card game. This formula uses the following facts:

- There are a total of $[C^P \cdot (C^P - 1)] / 6$ unique Sets in the entire deck
- Every card appears in $(C^P - 1) / 2$ Sets
- There are a total of $[N \cdot (N-1)] / 2$ Sets (both Crossover and Internal) that can be formed from the pairs within an N card pile of cards.

The total number of Internal Sets for N cards chosen from a C^P card deck, where there are P properties and C choices per property is then

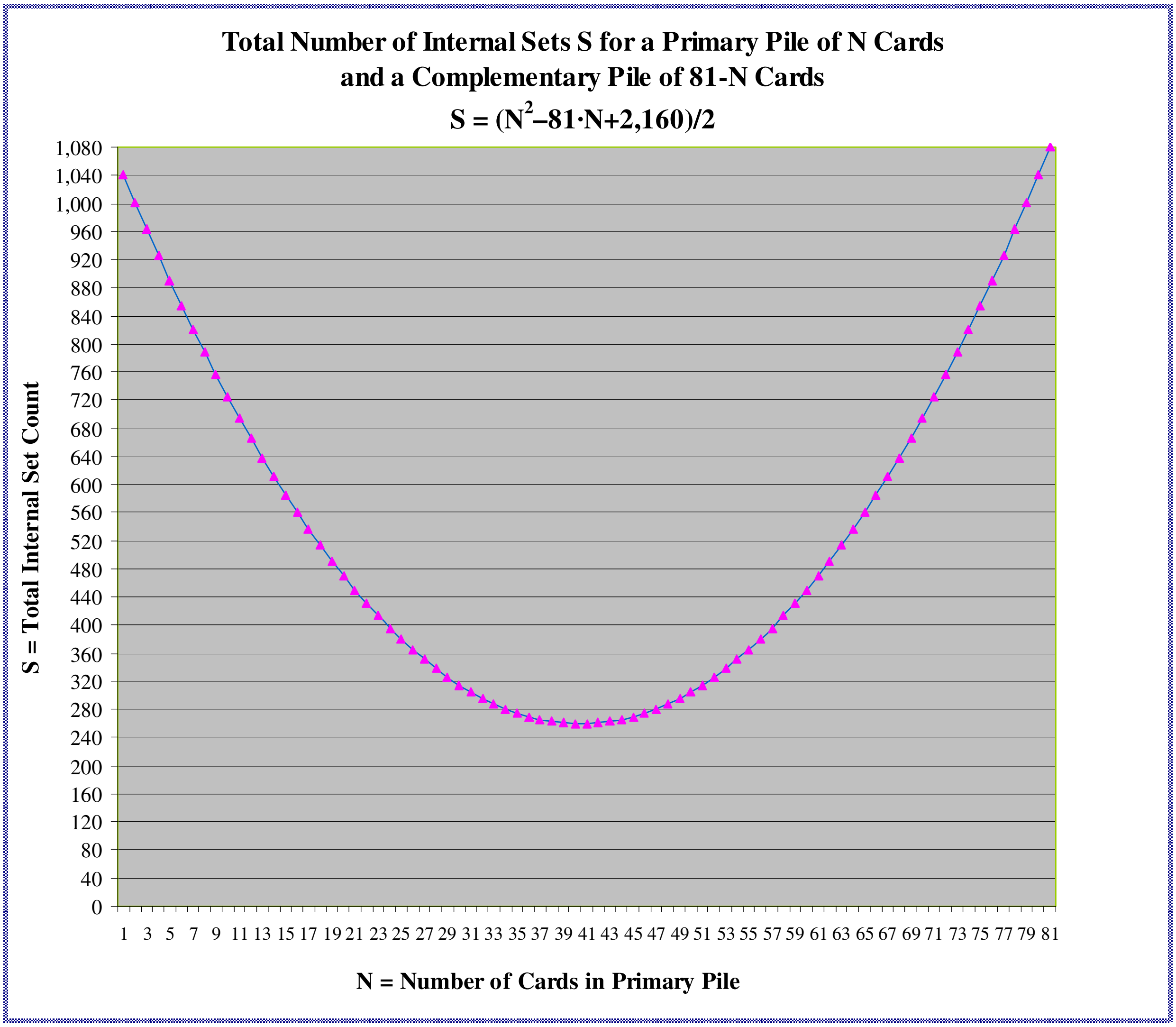
$$[C^P \cdot (C^P - 1)] / 6 - [(C^P - 1) / 2] \cdot N + [N \cdot (N-1)] / 2$$
$$= [(C^P - 1) \cdot (C^P - 3 \cdot N) / 6] + [N \cdot (N-1)] / 2$$

For example, applying this general formula to the card game SET, in which P = 4 and C = 3, we find that there are a total 470 total Internal Sets when N = 61. This is an important number, which we will come back to later.

For the case P = 5, it has been proven that the maximum number of cards without an Internal Set is 45 (for a detailed discussion, see, for example, the excellent paper, *The Card Game Set*, by Benjamin Lent Davis and Diane Maclagan). Five properties require an 81 x 3 = 243 card deck. Using the above formula, with P = 5, C = 3, and N = 45, we know that in order for 45 of the cards to have no Internal Set present, the other 198 cards must contain exactly 5,346 Internal Sets. I have verified that the illustration, found in the paper, does satisfy this requirement.

Now, if it were possible to develop a function for the maximum number of Internal Sets within a Primary Pile, it would be easy to determine the maximum number of cards that did not contain a Set. By calculating all of the values for the maximum Internal Set function and all of the values for the function of the total number of Internal Sets, and comparing the two results for each value of N, we could obtain a permissible range of values for which the Complementary Pile would contain all of the Internal Sets and the Primary Pile would be Setless. The maximum number of Internal Sets for one Pile always yields the minimum number of Internal Sets for the other Pile. We are interested in the cases where the minimum can be 0. The remainder of this paper will address the matter of determining the maximum Internal Set count.

The graph for the total number of Internal Sets, in an 81 card deck, when the deck is divided into a pile of N cards and a pile of 81−N cards, is shown below.



Set Blocking

Before proceeding with a discussion of the topic of the maximum number of Internal Sets within N cards, it is useful to gain an understanding of the strategy for blocking the formation of all Intracube and Intercube Sets. Blocking all Intracube Sets requires a minimum of 5 cards per cube, or 45 cards in total. But not just any 5 cards will do! In Formations 1 through 6 below, "P" represents a card in the Primary Pile, which is the Pile that we desire to be Setless, and "C" represents a card in the Complementary Pile, which is the set-blocking Pile. Formations 1 and 2 do not block all Sets in the Primary Pile, even though there are five cards present in the Complementary Pile. The Primary Pile forms one Set (shaded in yellow) in both of these formations.

Formation 1

C	C	C
C	C	P
P	P	P

Formation 2

P	C	C
C	C	P
P	P	C

Formation 3

C	P	P
C	P	P
C	C	C

Formation 4

C	P	C
P	C	P
C	P	C

Formation 5

P	C	P
C	C	C
P	C	P

Formation 6

P	C	P
P	C	P
C	C	C

One other difference to note between Formations 1 and 2 and Formations 3 through 6 is that the Complementary Piles in Formations 3 through 6 all form two Sets (shaded in green), which is the maximum number of Sets that can be formed from 5 cards.

The next concept to understand is that we also want to block Intercube Sets, and, as will be shown below, this requires picking more than 5 cards per cube for some of the cubes. The object is to produce a Complementary Pile with the minimum number of cards that blocks the formation of all Internal Sets within the Primary Pile. We can pick any one of Formations 3 through 6 as a starting point, but these are certainly not the only formations that we could use. In fact, because there are 12 lines per cube, 3 cards (points) in every line, and 3 additional lines that can be formed from any one card (point) in a particular line, there are (12 x 3 x 3) / 2 = 54 unique formations where five cards make two lines, out of the total of 126 ways to choose five cards from nine! However, it is unimportant which formation of five cards we choose, so long as we choose one that contains two Sets.

Let us choose Formation 4 and use this for Cube 1, as labeled in the illustration of all the cards found on page 6. If we choose the same formation for Cube 2, we are then faced with picking all nine cards in

Cube 3 in order to block Set formations for the eight cards in the Primary Pile. This is demonstrated in the following diagram.

Cube 1

C	P	C
P	C	P
C	P	C

Cube 2

C	P	C
P	C	P
C	P	C

Cube 3

2X	1X	2X
1X	4X	1X
1X	2X	2X

The notation in Cube 3 indicates all of the squares which must be blocked ("X" marks the spot!) in order to prevent the formation of Internal Sets for the Primary Pile of eight cards (four in Cube 1, four in Cube 2). The numbers in front of all of the "Xs" represent the number of Sets which are formed by these eight cards. In total, there are 4 x 4 = 16 Sets, and the middle card in Cube 3 is responsible for 4 of them. So if we used the above formation, we would have to add all nine cards in Cube 3 to the Complementary Pile in order to block all 16 Sets for the Primary Pile. However, keep in mind that we want to use as few of the cards in the Complementary Pile as possible.

If, instead, we use Formation 5 for Cube 2, we discover that only eight cards from Cube 3 are needed from the Complementary Pile (see the diagram below). Other Formations can be tried, but no less than eight cards are needed to block all 16 of the Intercube Sets for the Primary Pile.

Cube 1

C	P	C
P	C	P
C	P	C

Cube 2

P	C	P
C	C	C
P	C	P

Cube 3

2X	2X	2X
2X		2X
2X	2X	2X

It is important to recall the Set-forming cube combinations summarized at the top of page 8. The correct choices for Cubes 4 and 7, which form Sets with Cube 1, are the same ones used for Cubes 2 and 3, respectively. We now arrive at the following card selections, which demonstrate a Setless collection of 14 cards for the Primary Pile.

Cube 1

C	P	C
P	C	P
C	P	C

Cube 2

P	C	P
C	C	C
P	C	P

Cube 3

C	C	C
C	P	C
C	C	C

Cube 4

P	C	P
C	C	C
P	C	P

Cube 5

Cube 6

Cube 7

C	C	C
C	P	C
C	C	C

Cube 8

Cube 9

The choices for the remaining card formations in Cubes 5 through 9 are forced. The cards in Cubes 2, 4, and 9 form Sets, and because the patterns for Cubes 2 and 4 are identical, we must select all nine cards from Cube 9 in order to block the Sets formed by the cards from the Primary Pile. We can now pick 5 cards for the Complementary Pile in Cube 5, but if we choose the same formation as was used for Cube 2 or 4, we are forced to pick 9 cards from Cube 6. A better choice is the formation in Cube 1.

Our final choices for Cubes 6 and 8 require eight cards in each cube in the same formation used for Cubes 3 and 7. The complete diagram is shown below.

Cube 1

C	P	C
P	C	P
C	P	C

Cube 2

P	C	P
C	C	C
P	C	P

Cube 3

C	C	C
C	P	C
C	C	C

Cube 4

P	C	P
C	C	C
P	C	P

Cube 5

C	P	C
P	C	P
C	P	C

Cube 6

C	C	C
C	P	C
C	C	C

Cube 7

C	C	C
C	P	C
C	C	C

Cube 8

C	C	C
C	P	C
C	C	C

Cube 9

C	C	C
C	C	C
C	C	C

We observe that we need at least 61 cards in the Complementary Pile to block all Internal Sets for the Primary Pile. The number of Intracube and Intercube Sets are as follows:

Cube #	# of Cards	# of Intracube Sets
1	5	2
2	5	2
3	8	8
4	5	2
5	5	2
6	8	8
7	8	8
8	8	8
9	9	12
Total	**61**	**52**

Cube Combination			# of Cards			# of Intercube Sets
1	2	3	5	5	8	24
1	4	7	5	5	8	24
1	5	9	5	5	9	25
1	6	8	5	8	8	36
2	4	9	5	5	9	25
2	5	8	5	5	8	24
2	6	7	5	8	8	36
3	4	8	8	5	8	36
3	5	7	8	5	8	36
3	6	9	8	8	9	64
4	5	6	5	5	8	24
7	8	9	8	8	9	64
			Total			**418**

The total number of Internal Sets formed by the 61 card Complementary Pile is 418 + 52 = 470. Applying our earlier formula, we find that the total number of Internal Sets, when the deck is partitioned into two piles of 61 cards and 20 cards (we can substitute either N = 20 or N = 61 because of the symmetry), is:

$$(20^2 - 81\cdot 20 + 2{,}160) / 2 = 470$$

There are 470 total Internal Sets present, and we have demonstrated that all of them are accounted for by the 61 card pile alone.

Determining the Maximum Number of Internal Sets for N Cards

Developing a general formula for the maximum number of Internal Sets for N cards proved to be a formidable, and unproductive, task. Initially, I thought that the correct approach would be to maximize the Set count in a stepwise progression. Hereafter, I will refer to this as the Consecutive Maximization Method. The basic idea of this approach is to start with an arbitrary pair of cards, select the third card to make a Set with them, and continue on, turn by turn, each time choosing a card that maximized the number of Sets with unused pairs formed from the previous cards. Keep in mind that the ability to maximize the Internal Set count in the Complementary Pile results in a minimum Internal Set count for the Primary Pile.

One immediate difficulty was deciding whether it made a difference which card to choose when a tie occurred for the maximum Internal Set count. It was unclear whether a specific rule needed to be applied in such situations, but it is doubtful as to whether application of such a rule would have produced different results than presented here.

One possible progression through the cards was modeled using a Visual Basic program which applied the Consecutive Maximization Method under the following constraints:

1. The first two cards were chosen from different cubes in order to occupy the maximum number of cubes as quickly as possible. If the selection of the first two cards had been made from the same cube, this would have forced the selection of the next card to also be made from that cube in order to create the first Internal Set.
2. For turn 4, no new Internal Sets are possible because the first three cards were used to generate the first Internal Set, and there are no leftover pairs. Therefore, on this turn, and all turns in which a similar situation arose, the computer was directed to pick a card from a cube from which no card had yet been selected. In general, for turn $3^t + 1$, where $1 \le t \le P - 1$, and P = the number of card properties, it is impossible for new Internal Sets to be formed if Internal Set maximization occurred on the prior turn. Therefore, for these particular turns, the choice of cards is unrestricted.
3. When a tie occurred for the maximum Internal Set count on turn t, the "winner" was chosen to be the lowest available card number in a different cube from the cube associated with the

previously selected card which produced a Set. If this was not possible, the lowest numbered available card was chosen.

Although the Visual Basic program was written specifically to analyze the deck of cards for the case P = 4, I later decided to generalize the program to handle larger decks. Each card was defined by strings of 1s, 2s, and 3s to represent the three choices for each property. The program was written to automatically generate the deck of cards needed for analysis and the number of cubes necessary to display the solution, and to produce a table with a record of the results by turn and a diagram illustrating the solution that it found.

The following diagram illustrates the solution produced by the Visual Basic program for P = 4. It represents one possible progression through the cards such that the Internal Set count is maximized each turn, with the solution for each turn building on the cards selected from the prior turns.

The numbers indicate the turn number in which the card in the specified position was chosen. The card formation shown on page 6 for the entire 81 cards could be used to visualize which Sets are created by the card choice. There are many other paths to choose, but they will all yield the same results. For example, we could select our first nine cards from cube 1 and still produce 12 Sets.

Cube 1

1	10	12
28	30	32
36	42	44

Cube 2

2	13	11
35	38	40
29	31	33

Cube 3

3	14	15
34	39	41
37	43	45

Cube 4

4	16	20
46	50	52
60	70	72

Cube 5

6	21	17
58	66	68
47	51	53

Cube 6

7	22	24
59	67	69
61	71	73

Cube 7

5	18	23
48	54	56
64	78	80

Cube 8

8	25	19
62	74	76
49	55	57

Cube 9

9	26	27
63	75	77
65	79	81

Column 2 in Table 1 on the next two pages contains the maximum number of additional Internal Sets generated each turn using the Consecutive Maximization Method, and column 3 contains the cumulative counts. Column 4 contains the corrected counts, highlighted in green, which were determined by inspection. For example, it was discovered that 18 cards could produce 36 Internal Sets in two different ways (8, 5, 5 or 6, 6, 6). A graph of the maximum Internal Sets for each card count follows Table 1.

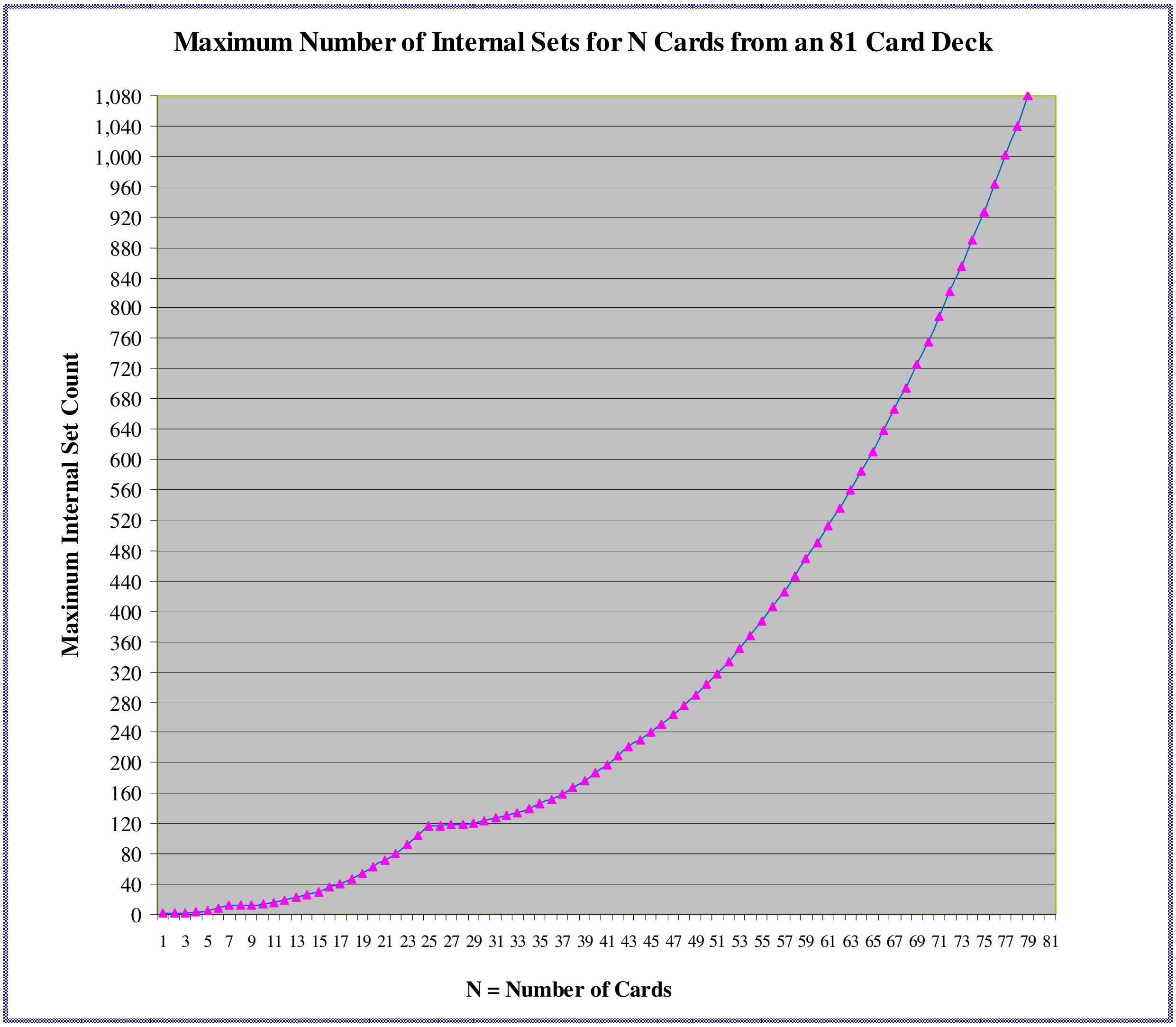
Table 1 – Maximum Internal Set Counts for N Cards

Number of Cards = N	Additional Internal Sets	Cumulative Maximum Internal Sets for N Cards	Corrected Maximum Internal Sets for N Cards[1]	Total Internal Set Count for N and 81-N Card Piles	Maximum Internal Set Count for 81-N Cards[1]	Minimum Number of Internal Sets for N Cards
3	1	1	1	963	963	0
4	0	1	1	926	926	0
5	1	2	2	890	890	0
6	1	3	3	855	855	0
7	2	5	5	821	821	0
8	3	8	8	788	788	0
9	4	12	12	756	756	0
10	0	12	12	725	725	0
11	1	13	13	695	695	0
12	1	14	14	666	666	0
13	2	16	16	638	638	0
14	3	19	19	611	611	0
15	4	23	23	585	585	0
16	3	26	26	560	560	0
17	4	30	30	536	536	0
18	5	35	36	513	513	0
19	6	41	41	491	491	0
20	6	47	47	470	470	0
21	7	54	54	450	447	3
22	8	62	62	431	425	6
23	9	71	71	413	406	7
24	10	81	81	396	388	8
25	11	92	92	380	369	11
26	12	104	104	365	351	14
27	13	117	117	351	334	17
28	0	117	117	338	318	20
29	1	118	118	326	303	23
30	1	119	119	315	289	26
31	2	121	121	305	276	29
32	3	124	124	296	264	32
33	4	128	128	288	252	36
34	3	131	131	281	241	40
35	4	135	135	275	231	44
36	5	140	140	270	222	48
37	6	146	146	266	209	57
38	6	152	152	263	197	66
39	7	159	159	261	186	75
40	8	167	167	260	176	84
41	9	176	176	260	167	93
42	10	186	186	261	159	102
43	11	197	197	263	152	111
44	12	209	209	266	146	120
45	13	222	222	270	140	130
46	9	231	231	275	135	140

Table 1 – Maximum Internal Set Counts for N Cards (continued)

Number of Cards = N	Additional Internal Sets	Cumulative Maximum Internal Sets for N Cards	Corrected Maximum Internal Sets for N Cards[1]	Total Internal Set Count for N and 81-N Card Piles	Maximum Internal Set Count for 81-N Cards[1]	Minimum Number of Internal Sets for N Cards
47	10	241	241	281	131	150
48	11	252	252	288	128	160
49	12	264	264	296	124	172
50	12	276	276	305	121	184
51	13	289	289	315	119	196
52	14	303	303	326	118	208
53	15	318	318	338	117	221
54	16	334	334	351	117	234
55	17	351	351	365	104	261
56	18	369	369	380	92	288
57	19	388	388	396	81	315
58	18	406	406	413	71	342
59	19	425	425	431	62	369
60	20	445	447	450	54	396
61	21	466	470	470	47	423
62	22	488	491	491	41	450
63	23	511	513	513	36	477
64	24	535	536	536	30	506
65	25	560	560	560	26	534
66	25	585	585	585	23	562
67	26	611	611	611	19	592
68	27	638	638	638	16	622
69	28	666	666	666	14	652
70	29	695	695	695	13	682
71	30	725	725	725	12	713
72	31	756	756	756	12	744
73	32	788	788	788	8	780
74	33	821	821	821	5	816
75	34	855	855	855	3	852
76	35	890	890	890	2	888
77	36	926	926	926	1	925
78	37	963	963	963	1	962
79	38	1,001	1,001	1,001	0	1,001
80	39	1,040	1,040	1,040	0	1,040
81	40	1,080	1,080	1,080	0	1,080

[1]Highlighted cases represent adjustments made from arrangements found to yield a higher count.


Maximum Number of Internal Sets for N Cards from an 81 Card Deck
Maximum Internal Set Count
1,080
1,040
1,000
960
920
880
840
800
760
720
680
640
600
560
520
480
440
400
360
320
280
240
200
160
120
80
40
0
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49 51 53 55 57 59 61 63 65 67 69 71 73 75 77 79 81
N = Number of Cards

Maximum Number of Internal Sets for Card Selection Restricted to 3 Cubes

In most cases, the Consecutive Maximization Method, with P = 4, produces results that likely cannot be improved upon. In fact, for N = 9 and N = 27, the results are verifiable because all pairs of cards are used to produce Internal Sets. With N = 9, there are 36 pairs and a maximum of 36/3 = 12 Internal Sets. With N = 27, there are $27 \cdot 26 / 2 = 351$ pairs and a maximum of 351/3 = 117 Internal Sets. Both of these results were produced by the Consecutive Maximization Method. For N = 4, N = 10, and N = 28, additional Internal Sets cannot be formed because there are no new available pairs.

The case of 36 Internal Sets being possible for N = 18 provides evidence that building off of prior solutions does not always work. For N = 61, we have already demonstrated that 470 Internal Sets can be constructed. The case of 447 Internal Sets for N = 60 was discovered by working backwards from the solution for N = 61. For N = 61, the Consecutive Maximization Method yields a solution of 466 maximum Internal Sets, which is only 4 less than the known solution of 470.

One approach that can be used to verify the maximum Internal Set counts is to decompose each of the card counts into all of the combinations of numbers that sum to the respective count and to determine how many Sets each arrangement is capable of producing. The question, “What is the maximum number of Internal Sets for N Cards?” can be restated as, “What is the optimal way to choose cards from each cube, so that the total number of cards chosen produces the maximum number of Internal Sets?” For example, 12 cards could be represented as 9 + 3 in 2 cubes, 6+3+3 in 3 cubes, 2+2+2+3+3 in 5 cubes, 1+1+1+3+3+3 in 6 cubes, et cetera. Of course, there are many other unique decompositions of the number 12 (69 to be exact). What we are interested in knowing is which arrangement produces the maximum number of Internal Sets for the 12 cards selected.

The idea is choose cards from as many cubes as required, but in a particular order, keeping in mind the line formations of the cubes. One method for cube selection is outlined in the table below.

Number of Cubes Required for Solution	Cubes Used for Card Selection
1	1
2	1 and 2
3	1, 2, and 3
4	1, 2, 3, and 4
5	1, 2, 3, 4, and 7
6	1, 2, 3, 4, 7, and 5
7	1, 2, 3, 4, 7, 5, and 6
8	1, 2, 3, 4, 7, 5, 6, and 8
9	1, 2, 3, 4, 7, 5, 6, 8, and 9

The cube numbering matches the labeling used on page 6. Note the following dependencies after the cards in the first two cubes are selected:

1. The cards chosen from cube 3 depend on the pairs of cards chosen in cubes 1 and 2.
2. The cards chosen from cube 7 depend on the pairs of cards chosen in cubes 1 and 4.
3. The cards chosen from cube 5 depend on pairs of cards chosen in cubes 3 and 7.
4. The cards chosen from cube 6 depend on the pairs of cards chosen in cubes 4 and 5 and cubes 2 and 7.
5. The cards chosen from cube 8 depend on the pairs of cards chosen in cubes 2 and 5, cubes 3 and 4, and cubes 1 and 6.
6. The cards chosen from cube 9 depend on the pairs of cards chosen in cubes 1 and 5, cubes 2 and 4, cubes 3 and 6, and cubes 7 and 8.

The task of evaluating the total number of Internal Sets formed becomes rather complicated once you get beyond six cubes because of the multiple constraints that must be satisfied. The problem we are confronted with is similar to solving multiple equations in multiple unknowns. The "variables" or "unknowns" in this case are the cube configurations which block the formation of all Internal Sets for the Primary Pile and which maximize the Internal Set count for the Complementary Pile.

In some cases, it may not be possible to maximize the Internal Set count, in every direction, with a particular configuration. For example, we already know that 36 Internal Sets can be produced with an 18 card collection. Because it is possible to pick 2 cards in each of three cubes and produce 4 Internal Sets from this combination, we might wrongly conclude that we can select 2 cards from each of the nine cubes to get 18 cards and produce 4 x 9 = 36 Internal Sets. However, it can be demonstrated that it is only possible to form 34 Internal Sets with this arrangement.

Another complication with trying to find cube combinations that yield the maximum Internal Set count is that there are a total of 48,609 different representations for the sums 3 through 81! However, only arrangements where there are at least five cards in a cube have to be analyzed because five cards is the minimum number of cards required to block all Intracube Sets. If we restrict our attention to these configurations, there are 9,074 cube combinations to consider.

I decided to take the first step by determining the maximum number of Internal Sets that can be produced by restricting the card selection to 3 cubes, which form the building blocks of all combinations. The maximums are summarized in Table 2 on the next three pages. The table is sorted with the Set counts in descending order by total card count and cube card combination. The optimal solutions for each combination are highlighted, but it should be noted that all arrangements, with the specified number of cards, will not always work. However, there is at least one arrangement that will produce the maximum Internal Set count indicated in the table. It is interesting to note that neither of the possible solutions for the maximum Internal Set count for N = 18 involve 3 or 9 card selections for one of the three cubes. Finally, for N ≥ 23, any of the possible combinations will produce the maximum Internal Set count.

Table 2 - Maximum Internal Sets for 3 Cube Card Selections

Sum	3 Cube Card Combination	Maximum Internal Set Count	Sum	3 Cube Card Combination	Maximum Internal Set Count
3	111	1	11	911	13
4	211	1	11	533	13
5	311	2	11	443	13
5	221	2	11	821	10
6	321	3	11	632	10
6	222	3	11	542	10
6	411	2	11	731	9
7	331	5	11	722	9
7	322	5	11	551	9
7	511	3	11	641	8
7	421	3	12	921	14
8	332	8	12	633	14
8	431	5	12	543	14
8	422	5	12	444	14
8	611	4	12	831	12
8	521	4	12	822	12
9	333	12	12	732	12
9	432	8	12	552	12
9	711	6	12	642	11
9	531	6	12	741	10
9	522	6	12	651	10
9	441	6	13	931	16
9	621	5	13	922	16
10	433	12	13	733	16
10	811	9	13	643	16
10	532	9	13	553	16
10	442	9	13	544	16
10	721	7	13	832	15
10	631	7	13	742	14
10	622	7	13	652	14
10	541	7	13	841	13
			13	751	12
			13	661	12

Table 2 - Maximum Internal Sets for 3 Cube Card Selections (continued)

Sum	3 Cube Card Combination	Maximum Internal Set Count
14	932	19
14	833	19
14	743	19
14	554	19
14	653	18
14	644	18
14	941	17
14	842	17
14	752	17
14	662	16
14	851	15
14	761	14
15	933	23
15	744	23
15	663	23
15	555	23
15	843	22
15	654	22
15	942	21
15	753	21
15	852	20
15	951	19
15	762	19
15	861	17
15	771	17
16	943	26
16	844	26
16	763	26
16	754	26
16	664	26
16	655	26
16	853	25
16	952	24
16	862	23
16	772	23
16	961	21
16	871	20

Sum	3 Cube Card Combination	Maximum Internal Set Count
17	953	30
17	944	30
17	854	30
17	773	30
17	764	30
17	755	30
17	665	30
17	863	29
17	962	27
17	872	27
17	971	24
17	881	24
18	855	36
18	666	36
18	954	35
18	864	35
18	774	35
18	765	35
18	963	34
18	873	34
18	972	31
18	882	31
18	981	28
19	955	41
19	865	41
19	775	41
19	766	41
19	964	40
19	874	40
19	973	39
19	883	39
19	982	36
19	991	33

Table 2 - Maximum Internal Sets for 3 Cube Card Selections (continued)

Sum	3 Cube Card Combination	Maximum Internal Set Count
20	965	47
20	875	47
20	866	47
20	776	47
20	974	46
20	884	46
20	983	45
20	992	42
21	975	54
21	966	54
21	885	54
21	876	54
21	777	54
21	984	53
21	993	52
22	985	62
22	976	62
22	886	62
22	877	62
22	994	61
23	995	71
23	986	71
23	977	71
23	887	71
24	996	81
24	987	81
24	888	81
25	997	92
25	988	92
26	998	104
27	999	117

The Maximum Number of Internal Sets for N Cards and Number of Properties P = 4

This section will discuss a general strategy to employ in order to maximize the Internal Set count for each particular value of N, $3 \leq N \leq 81$. Appendix I contains illustrations of some of the possible card selections that can be made to produce the maximum number of Internal Sets for a particular value of N. These illustrations were produced by inspection of the cards and differ from the solutions that can be derived from the diagram on page 19. The Internal Set counts agree, except in the cases noted below.

The first thing to recognize is that the strategy for picking the first 45 cards is different from the strategy for picking the last 36 cards. As noted earlier, there are a few check points throughout the process. We already know the maximum Internal Set counts for N = 9 and N = 27 are 12 and 117, respectively. In determining the maximum for the first 27 cards, there is no advantage in selecting cards from more than 3 cubes because at least 5 cards are required, from each of the nine cubes, to block all Intracube Set formations. So whether we pick the top row of 3 cards from each of the nine cubes, or pick 9 cards from 3 cubes, the outcome will be the same. Therefore, for $N \leq 27$, we restrict our attention to selecting cards from cubes 1, 2, and 3. Note that any other line-forming cube combination could be used, but we will use this one for simplicity.

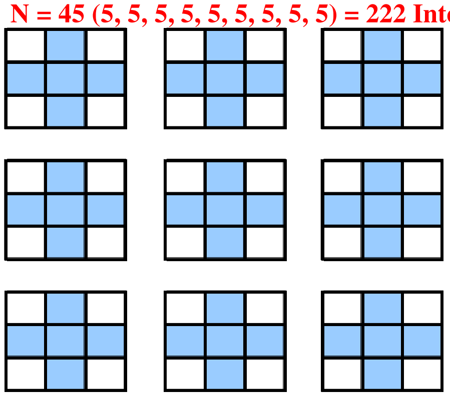
Once we have chosen our first 9 cards, we are out of pairs to form any more Sets. Therefore, the 10th card we select will not increase the Internal Set count. We reach a maximum Internal Set count of 12 for both N = 9 and N = 10. Our first critical position arises at N = 18. The optimal solution calls for either a 3 cube combination of 8, 5, 5 or 6, 6, 6, but we are unable to reach either of these positions from the ones shown in the preceding diagram found on page 36. Therefore, we must start over and select cards so that we can form 36 Internal Sets from the 18 cards.

We can build off the optimal solution at N = 18 and proceed with the Consecutive Maximization Method, beginning with N = 19. However, when we reach N = 27, we will have used up all pairs and be in the same position that we were in after N = 9. The best we can do for cards 28 through 45 is to form Intercube Sets with either cube 1, 2, or 3. We will follow the cube selection process outlined on page 23 and create an "L" formation by using cubes 4 and 7 for our next card selections. When we reach N = 45 cards, the maximum Internal Set count will be 117 (first 27 cards) + 81 (new Intercube Sets) + 24 (new Intracube Sets) = 222. Therefore, the maximum Internal Set count for N between 28 and 45 will fall in the range of 117 and 222.

It is worth noting that, if we use one of the 5-card blocking formations in each of the nine cubes, we can also produce 222 Internal Sets for N = 45. One possible choice is illustrated on the next page.

Although we have blocked all Intracube Sets, there are still 48 Intercube Sets formed by the Primary Pile of 36 cards remaining.

N = 45 (5, 5, 5, 5, 5, 5, 5, 5, 5) = 222 Internal Sets



Beginning with card 46, our strategy shifts to choosing cards from all of the remaining cubes and substituting the 5-card block at the appropriate time. There are four remaining cubes for which no cards have yet been chosen, so it will take at least four more turns until cards have been picked from all nine cubes. We can maximize the Internal Set count by successively picking the optimal card from cubes 5, 6, 8, or 9. However, we reach another critical point at N = 59.

Diagram A

N = 59 (9, 9, 9, 9, 9, 4, 4, 3, 3) = 425 Internal Sets

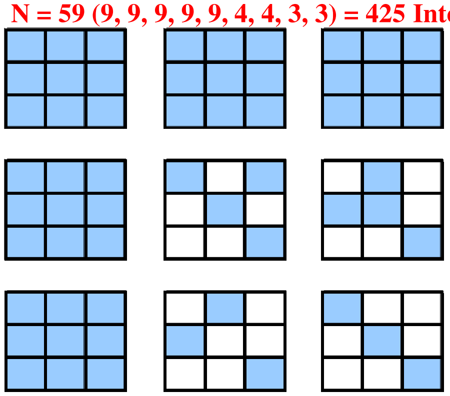
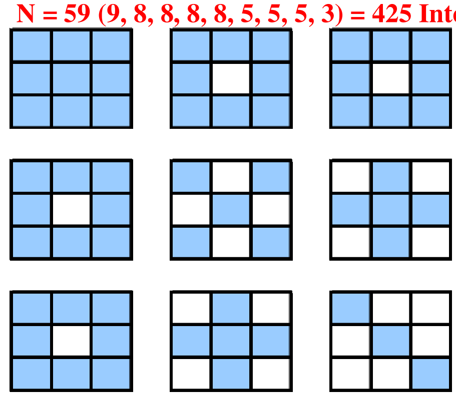


Diagram B

N = 59 (9, 8, 8, 8, 8, 5, 5, 5, 3) = 425 Internal Sets



In Diagram A, the card in the bottom right corner in both cubes 6 and 8 does not fit the pattern of the known solution at N = 61. Also, there are two suboptimal configurations for the sum 22 using the triple of cards 9,9,4. This triple generates 61 Internal Sets instead of the maximum of 62. We need to determine if there is a benefit to exchanging our card choices for the middle card in cubes 2, 3, 4, and 7 and making new card choices for cubes 6 and 8. However, as seen by the configuration in Diagram B, switching cards does not do us any good yet because we still produce only 425 Internal Sets.

A lesson to be learned from this example is that the presence of a suboptimal 3 cube configuration alone does not prevent us from maximizing the Internal Set count.

For N = 60, a different picture emerges. We can choose another card from either of the above patterns. The two resulting solutions for N = 60 are as follows:

Diagram A

N = 60 (9, 9, 9, 9, 9, 4, 4, 4, 3) = 445 Internal Sets

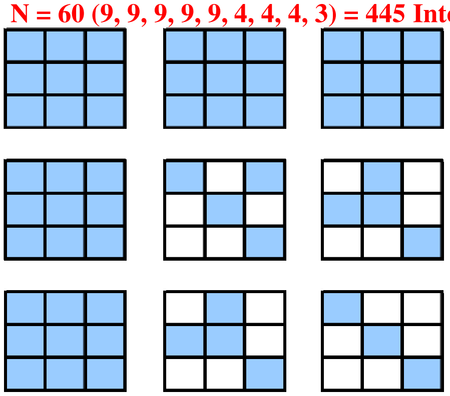
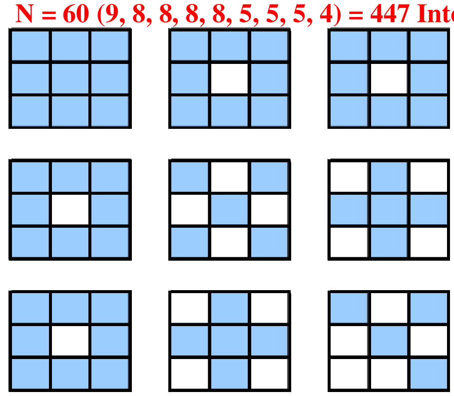


Diagram B

N = 60 (9, 8, 8, 8, 8, 5, 5, 5, 4) = 447 Internal Sets



As shown in Diagram B, the re-selection of cards at N = 60 to create the 5-card block yields two more Internal Sets than what can be accomplished with the configuration in Diagram A. This outcome suggests that a good strategy is to replace suboptimal 3 cube configurations when they occur and make our next card choice from the improved arrangement.

For our 61st card, we complete the "X" formation in cube 9 by selecting the card in the lower left corner of this cube. For subsequent turns, any of the remaining selections will maximize the Internal Set count.

Patterns from the Consecutive Maximization Method

Three interesting patterns emerged from application of the Consecutive Maximization Method. The first one is that the solution for P = 3, 4, 5, and 6 was consistently of the form 2^P.

The second pattern, which was not immediately apparent, is related to the difference between the known solution for the maximum number of cards without an Internal Set and the solution predicted by examining the output from the Consecutive Maximization Method. The table below shows a comparison of the results.

P = Number of Properties	Maximum Number of Setless Cards		Difference Between Actual and Predicted Solution	Predicted Error In Modeled Result
	Known	Modeled		
3	9	$8 = 2^3$	1	1
4	20	$16 = 2^4$	4	4
5	45	$32 = 2^5$	13	13
6	112, 113, or 114	$64 = 2^6$	48, 49, or 50	40

For P = 3, 4, and 5, the differences between the modeled solution and the known solution for the maximum number of Setless cards matched the values from the formula

$$(3^{P-2} - 1) / 2 = \text{number of Intercube line formations per cube}$$

The last formula results from the following facts for a deck with P properties.

1. There are 3^P cards in the deck.
2. It takes 3^{P-2} cubes to represent the Sets for the 3^P cards.
3. There are $[3^{P-2} \cdot (3^{P-2} - 1)] / 2$ pairs of cube line formations that are the basis for all Intercube Sets.

The number of Intercube line formations per cube is then $(3^{P-2} - 1) / 2$, which is the pattern produced by the solutions for P = 3, 4, and 5.

The third and final pattern that emerged is related to the difference between the maximum number of Internal Sets produced by the Complementary Pile, for the known solutions, and the modeled maximum number of Internal Sets produced by the Current Maximization Method. The results are summarized in the table below.

P = Number of Properties	Number of Cards in Deck	Known Maximum Number of Setless Cards	Number of Cards in Complementary Pile	Maximum Number of Internal Sets for Complementary Pile		Difference
				Known	Modeled	
3	27	9	18	36	35	1
4	81	20	61	470	466	4
5	243	45	198	5,333	5,346	13

The above pattern for the differences also fits the formula $(3^{P-2} - 1) / 2$. For N = 3, 4, and 5, the maximum number of Setless cards, for P properties, can be expressed by the formula,

$$2^P + [(3^{P-2} - 1)] / 2 = [2^{P+1} + (3^{P-2} - 1)] / 2$$

Although a solution to the problem of the maximum number of cards that can occur without an Internal Set has not yet been solved for the deck with 6 properties, the answer has been demonstrated, using higher level mathematics, to be either N = 112, N = 113, or N = 114. Therefore, the actual card count difference between the modeled solution of 64 and this possible range of solutions has to be 48, 49, or 50, all of which are different from the result of 40 predicted by the formula $(3^{P-2} - 1) / 2$. For P = 6, the above formula yields the result 104 cards, which does not match any of the possible solutions.

Using the general formula developed earlier, we can determine the number of Internal Sets that must be present in the Complementary Piles in order for the proposed maximum to be the correct solution. These values are summarized on the next page.

Internal Set Count with $3^6 = 729$ Card Deck and $112 \leq N \leq 114$

Possible Maximum Number of Setless Cards N	Number of Cards in Complementary Pile	Number of Internal Sets Complementary Pile Must Form if N is Solution	Number of Internal Sets Modeled for Complementary Pile	Difference
112	617	53,900	53,820	80
113	616	53,648	53,565	83
114	615	53,397	53,311	86

Once again, the possible differences do not fit the preceding pattern. It remains to be shown which one of the above values is the correct solution. I have included the diagrams produced by the Consecutive Maximization Method for P = 3, 4, 5, and 6 as well as the known solutions for P = 3, 4, and 5. The Consecutive Maximization Method appears to have some merit and may be worth investigating further. For example, although the Consecutive Maximization Method does not yield the optimal result for the maximum number of Setless cards, it does provide a lower bound. For example, for P = 6, we know that the maximum cannot be lower than 64 cards.

If the solution produced by the Consecutive Maximization Method can be shown to always be of the form 2^P, where P = number of properties, and the correction adjustment can also be expressed as a formula, then we can know, by a quick calculation, the maximum number of cards for any size deck of cards, where there are P properties and the number of features for each property is three. The only drawback would be the inability to automatically translate the answer into a diagram.

Conclusion

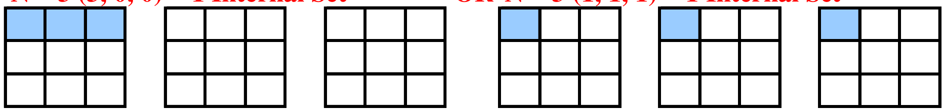
The mathematically inclined reader is encouraged to consider whether there is a more rigorous and/or direct method for determining the maximum number of Internal Sets, given N cards have been chosen, and to study the patterns set forth in this paper. This effort may lead to discovering a general solution for the maximum SET problem for a larger deck of cards in which the rules of SET still apply. As of this paper, solutions exist for up to five properties.

I strongly encourage math teachers to use the game SET in their classroom to engage students by further exploring some of the topics discussed in this paper as well as other topics, such as the probabilities of not getting a SET when N cards are chosen from the deck. For example, students could identify which formations of 12 cards do not produce any Sets and analyze how many ways the various combinations can occur. Mathematics is best learned by the use of visual illustrations, and SET provides many opportunities to teach key concepts from probability and statistics in this manner.

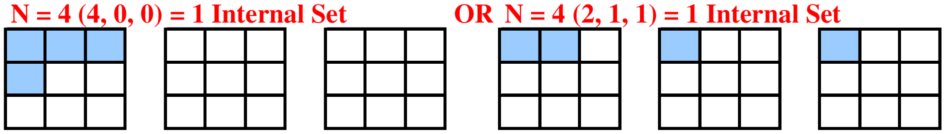
Finally, I wish to thank my good friend and colleague, Cameron Williams, for putting up with my many discourses on SET, for providing his own unique analysis, and for taking the time to read this paper. Cameron’s endless energy and keen ability to reason quickly through complex problems is a constant source of inspiration. I also wish to thank the creators of SET for creating a very intriguing game with extensive mathematical applications and welcome feedback from readers who may have fresh insights into the ideas which have been presented. In particular, I would be interested to hear from anyone who is able to produce a general formula for the maximum number of Internal Sets for a given number of cards. My e-mail address can be found on the cover page of this document.

APPENDIX I: MAXIMUM INTERNAL SET COUNTS

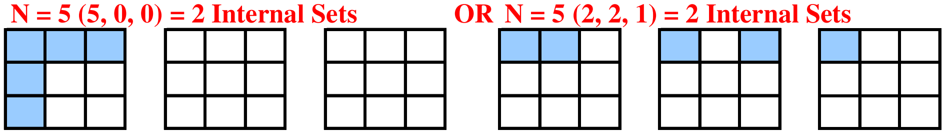
N = 3 (3, 0, 0) = 1 Internal Set OR N = 3 (1, 1, 1) = 1 Internal Set



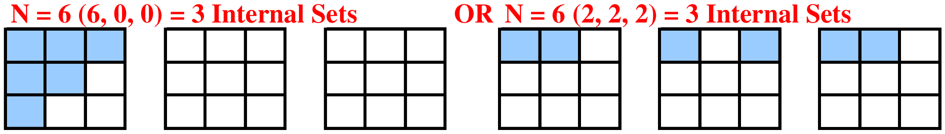
N = 4 (4, 0, 0) = 1 Internal Set OR N = 4 (2, 1, 1) = 1 Internal Set



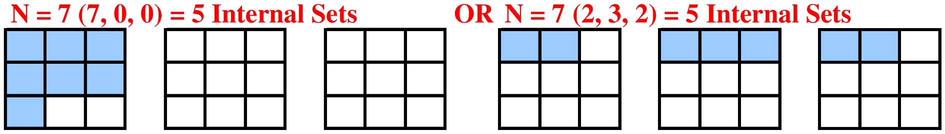
N = 5 (5, 0, 0) = 2 Internal Sets OR N = 5 (2, 2, 1) = 2 Internal Sets



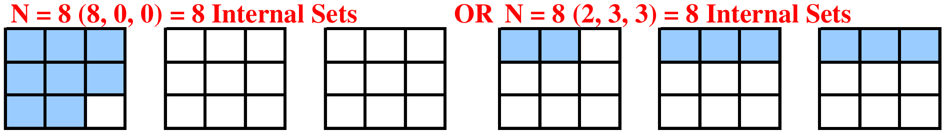
N = 6 (6, 0, 0) = 3 Internal Sets OR N = 6 (2, 2, 2) = 3 Internal Sets



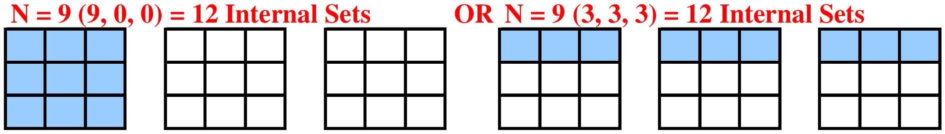
N = 7 (7, 0, 0) = 5 Internal Sets OR N = 7 (2, 3, 2) = 5 Internal Sets



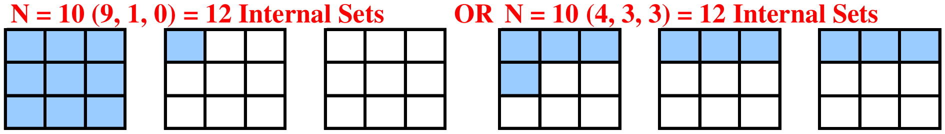
N = 8 (8, 0, 0) = 8 Internal Sets OR N = 8 (2, 3, 3) = 8 Internal Sets



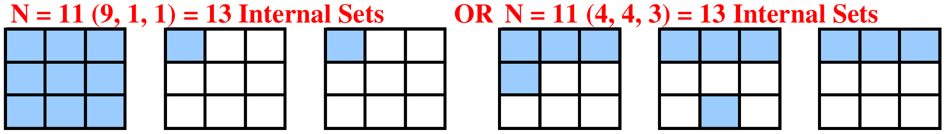
N = 9 (9, 0, 0) = 12 Internal Sets OR N = 9 (3, 3, 3) = 12 Internal Sets



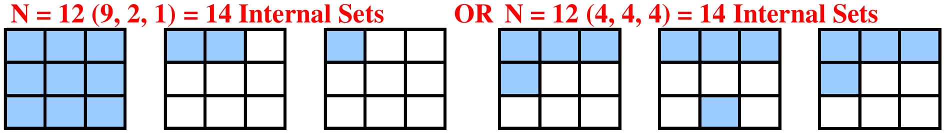
N = 10 (9, 1, 0) = 12 Internal Sets OR N = 10 (4, 3, 3) = 12 Internal Sets



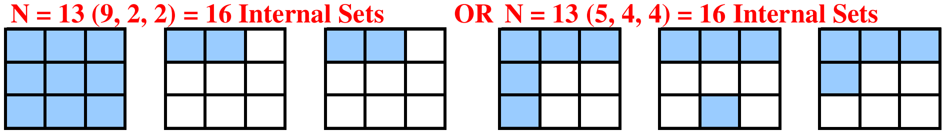
N = 11 (9, 1, 1) = 13 Internal Sets OR N = 11 (4, 4, 3) = 13 Internal Sets



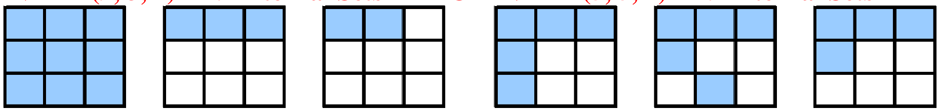
N = 12 (9, 2, 1) = 14 Internal Sets OR N = 12 (4, 4, 4) = 14 Internal Sets



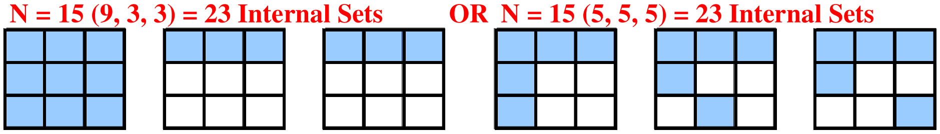
N = 13 (9, 2, 2) = 16 Internal Sets OR N = 13 (5, 4, 4) = 16 Internal Sets



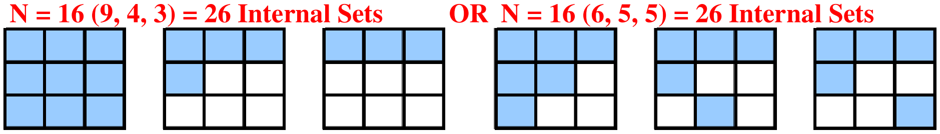
N = 14 (9, 3, 2) = 19 Internal Sets OR N = 14 (5, 5, 4) = 19 Internal Sets



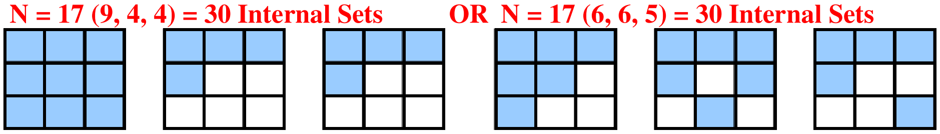
N = 15 (9, 3, 3) = 23 Internal Sets OR N = 15 (5, 5, 5) = 23 Internal Sets



N = 16 (9, 4, 3) = 26 Internal Sets OR N = 16 (6, 5, 5) = 26 Internal Sets

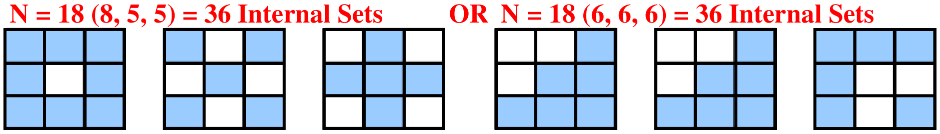


N = 17 (9, 4, 4) = 30 Internal Sets OR N = 17 (6, 6, 5) = 30 Internal Sets

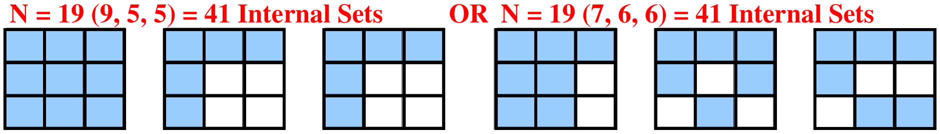


The next two patterns cannot be created from the preceding ones for N = 17.

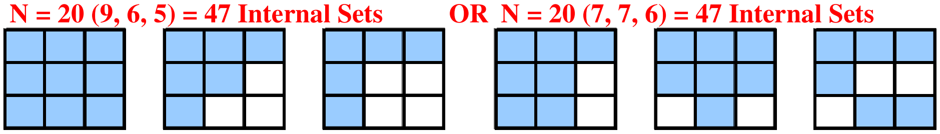
N = 18 (8, 5, 5) = 36 Internal Sets OR N = 18 (6, 6, 6) = 36 Internal Sets



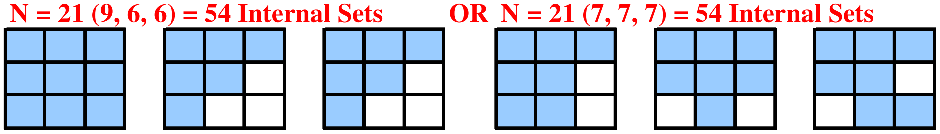
N = 19 (9, 5, 5) = 41 Internal Sets OR N = 19 (7, 6, 6) = 41 Internal Sets



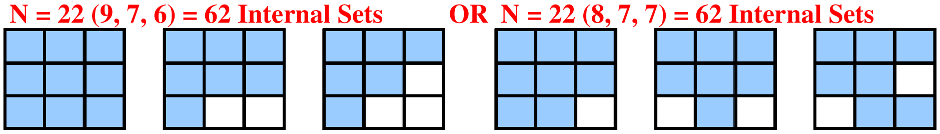
N = 20 (9, 6, 5) = 47 Internal Sets OR N = 20 (7, 7, 6) = 47 Internal Sets



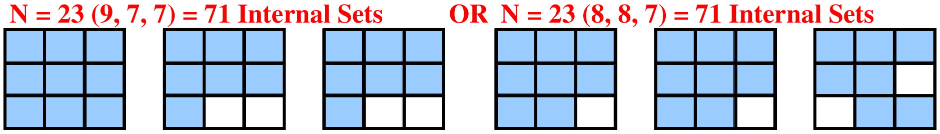
N = 21 (9, 6, 6) = 54 Internal Sets OR N = 21 (7, 7, 7) = 54 Internal Sets



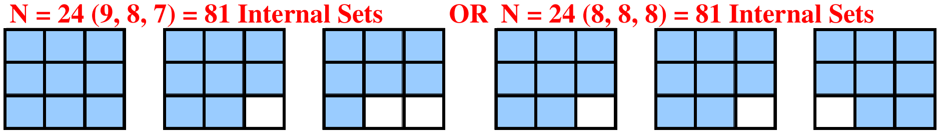
N = 22 (9, 7, 6) = 62 Internal Sets OR N = 22 (8, 7, 7) = 62 Internal Sets



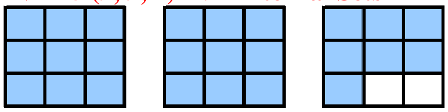
N = 23 (9, 7, 7) = 71 Internal Sets OR N = 23 (8, 8, 7) = 71 Internal Sets



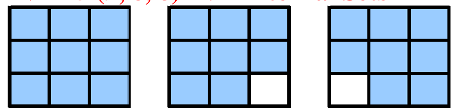
N = 24 (9, 8, 7) = 81 Internal Sets OR N = 24 (8, 8, 8) = 81 Internal Sets



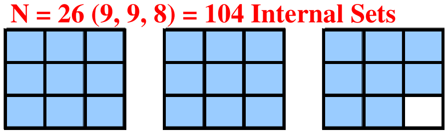
N = 25 (9, 9, 7) = 92 Internal Sets



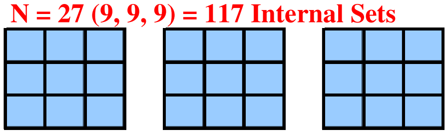
OR N = 25 (9, 8, 8) = 92 Internal Sets



N = 26 (9, 9, 8) = 104 Internal Sets

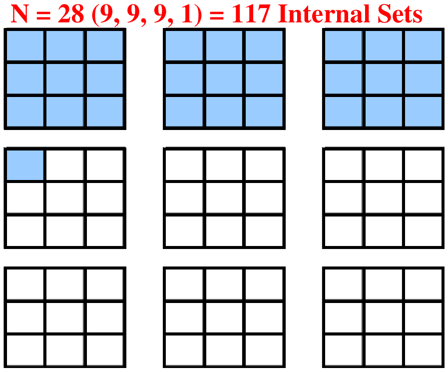


N = 27 (9, 9, 9) = 117 Internal Sets

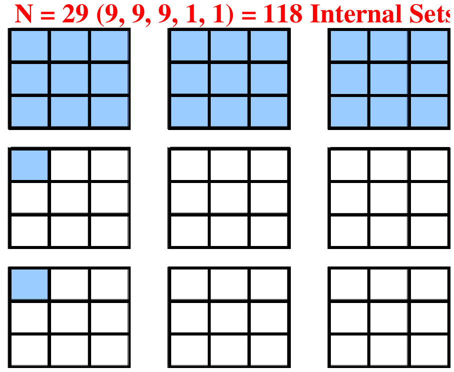


We have used up all of our pairs, so it's time to create a new line of attack! Cubes 4 and 7 will now be used.

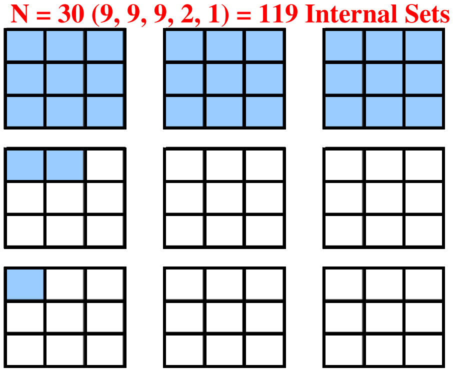
N = 28 (9, 9, 9, 1) = 117 Internal Sets



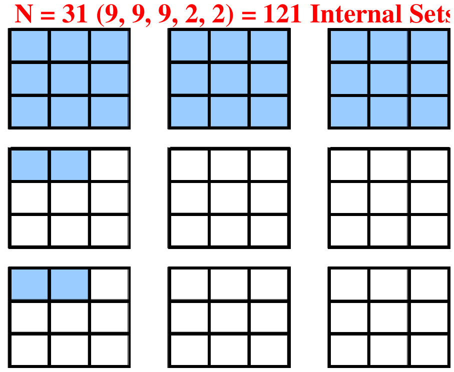
N = 29 (9, 9, 9, 1, 1) = 118 Internal Sets



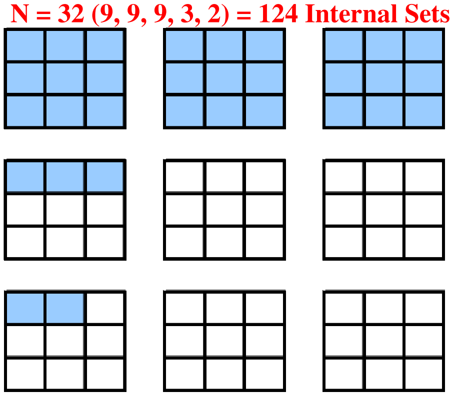
N = 30 (9, 9, 9, 2, 1) = 119 Internal Sets



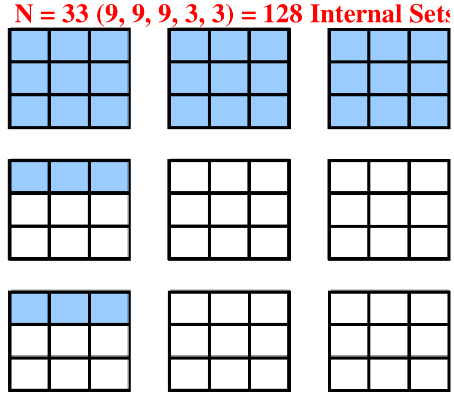
N = 31 (9, 9, 9, 2, 2) = 121 Internal Sets



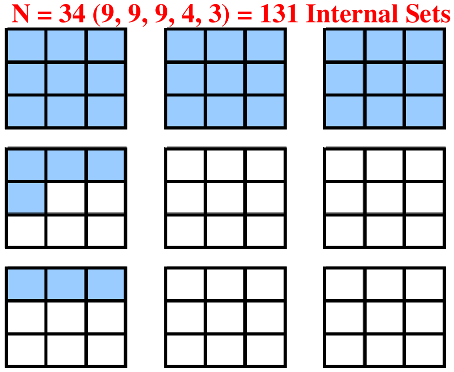
N = 32 (9, 9, 9, 3, 2) = 124 Internal Sets



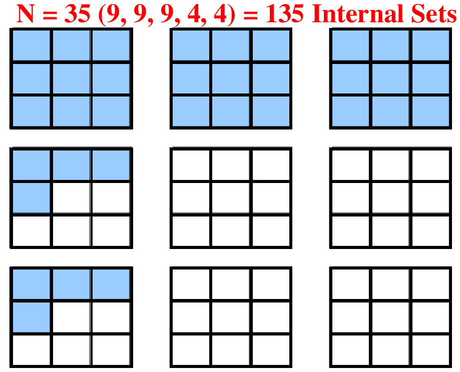
N = 33 (9, 9, 9, 3, 3) = 128 Internal Sets



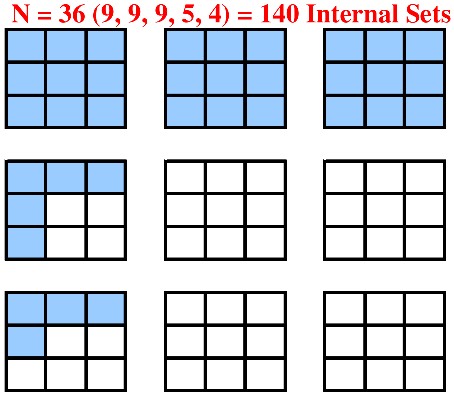
N = 34 (9, 9, 9, 4, 3) = 131 Internal Sets



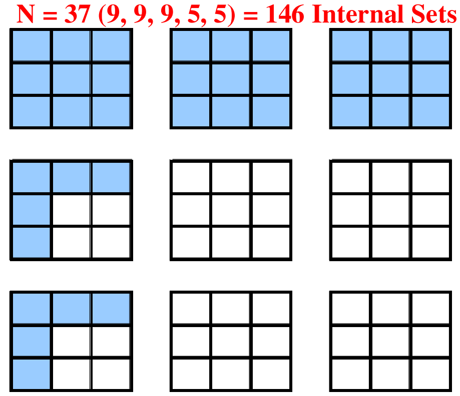
N = 35 (9, 9, 9, 4, 4) = 135 Internal Sets



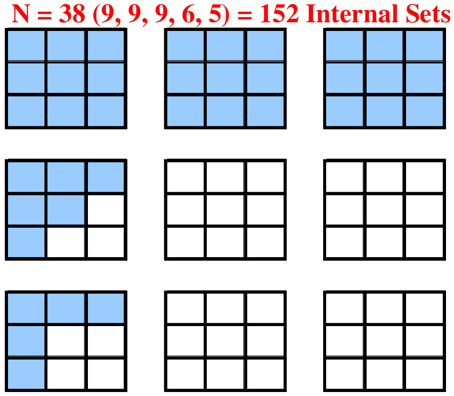
N = 36 (9, 9, 9, 5, 4) = 140 Internal Sets



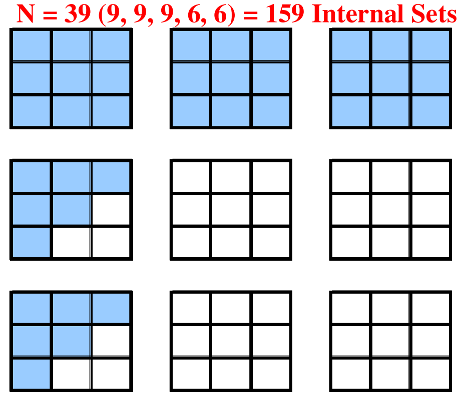
N = 37 (9, 9, 9, 5, 5) = 146 Internal Sets



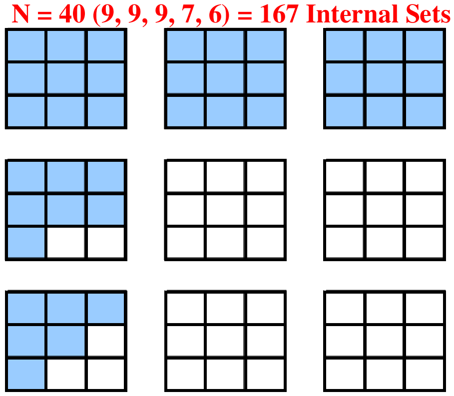
N = 38 (9, 9, 9, 6, 5) = 152 Internal Sets



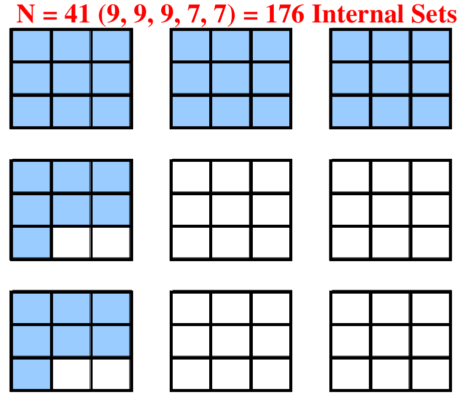
N = 39 (9, 9, 9, 6, 6) = 159 Internal Sets



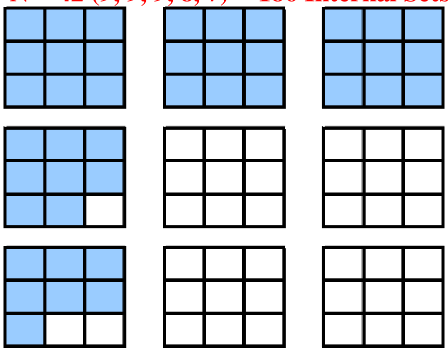
N = 40 (9, 9, 9, 7, 6) = 167 Internal Sets



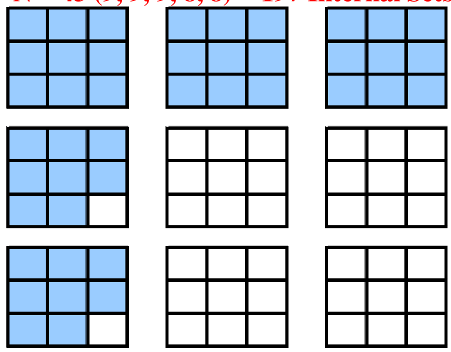
N = 41 (9, 9, 9, 7, 7) = 176 Internal Sets



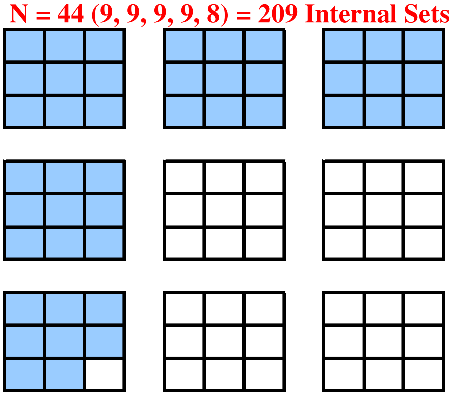
N = 42 (9, 9, 9, 8, 7) = 186 Internal Sets



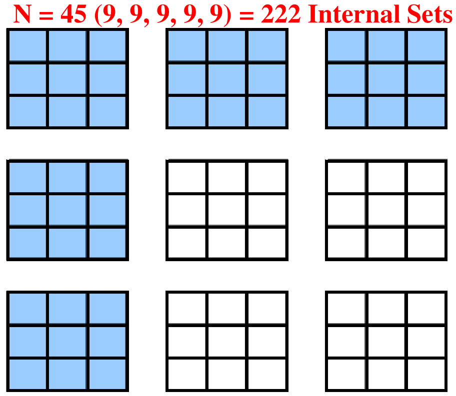
N = 43 (9, 9, 9, 8, 8) = 197 Internal Sets



N = 44 (9, 9, 9, 9, 8) = 209 Internal Sets

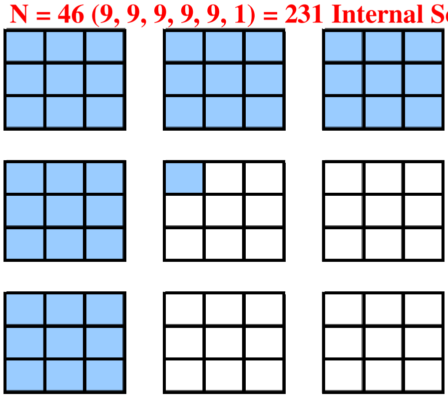


N = 45 (9, 9, 9, 9, 9) = 222 Internal Sets

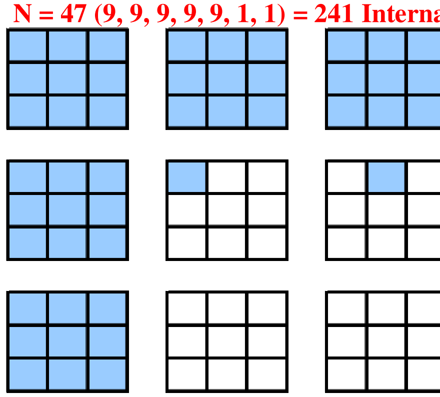


We have reached another critical point. It's time to create multiple lines of attack!

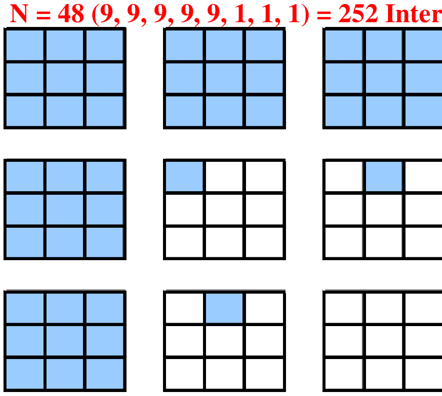
N = 46 (9, 9, 9, 9, 9, 1) = 231 Internal Sets



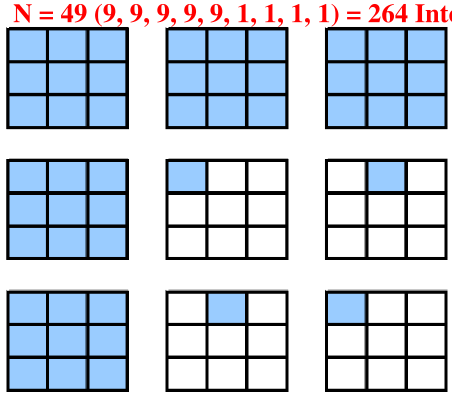
N = 47 (9, 9, 9, 9, 9, 1, 1) = 241 Internal Sets



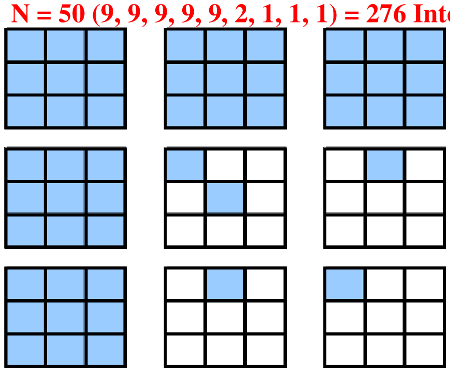
N = 48 (9, 9, 9, 9, 9, 1, 1, 1) = 252 Internal Sets



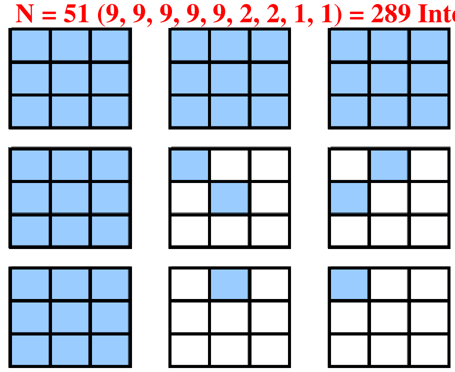
N = 49 (9, 9, 9, 9, 9, 1, 1, 1, 1) = 264 Internal Sets



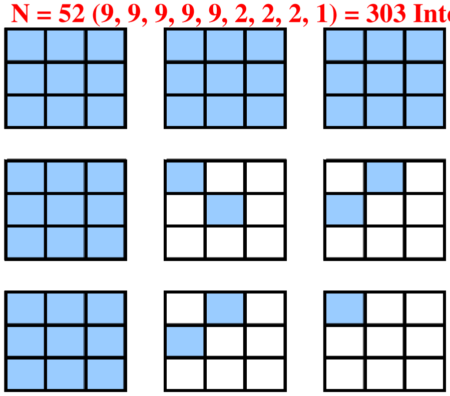
N = 50 (9, 9, 9, 9, 9, 9, 2, 1, 1, 1) = 276 Internal Sets



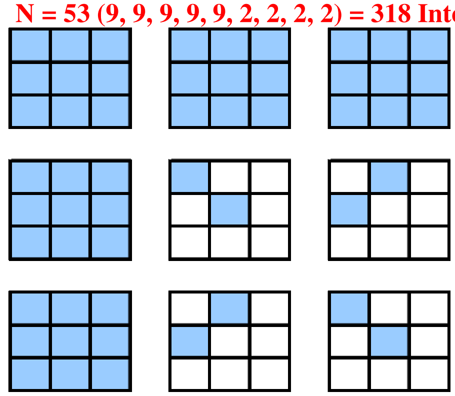
N = 51 (9, 9, 9, 9, 9, 9, 2, 2, 1, 1) = 289 Internal Sets



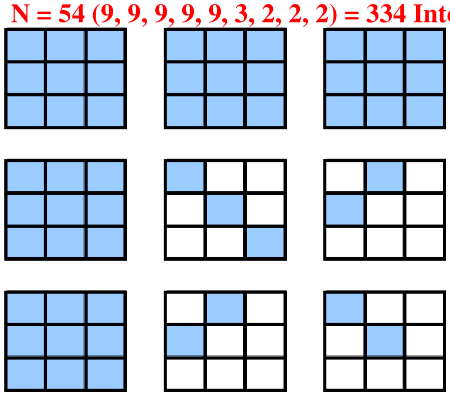
N = 52 (9, 9, 9, 9, 9, 9, 2, 2, 2, 1) = 303 Internal Sets



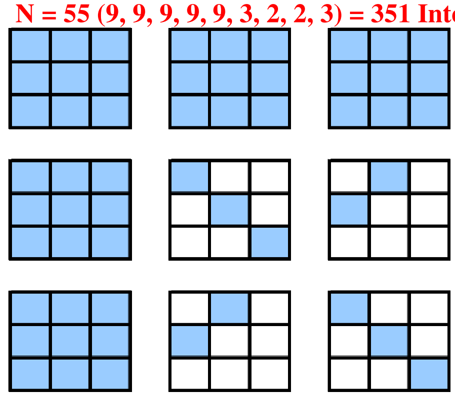
N = 53 (9, 9, 9, 9, 9, 9, 2, 2, 2, 2) = 318 Internal Sets



N = 54 (9, 9, 9, 9, 9, 9, 3, 2, 2, 2) = 334 Internal Sets

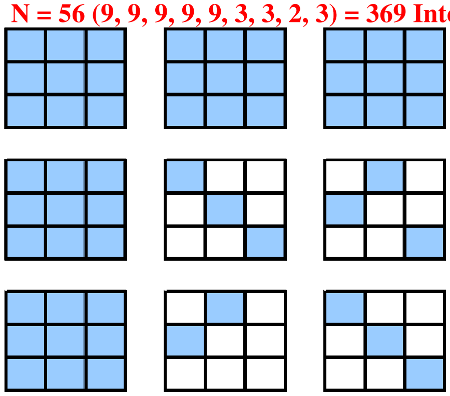


N = 55 (9, 9, 9, 9, 9, 9, 3, 2, 2, 3) = 351 Internal Sets

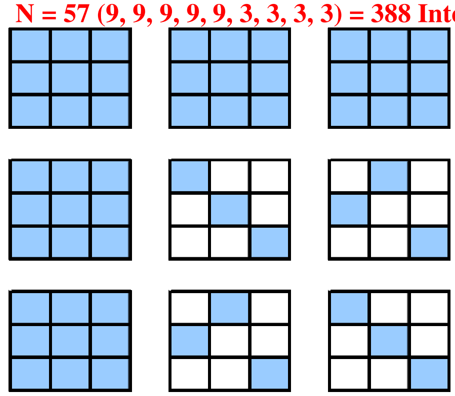


For N = 54, if we had instead chosen 9 cards from cube 5 and no cards from cubes 6, 8, and 9, we would only produce 315 Sets!

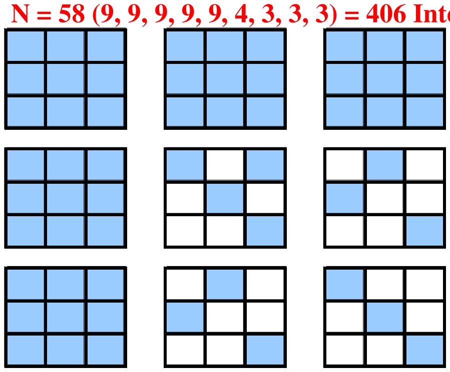
N = 56 (9, 9, 9, 9, 9, 9, 3, 3, 2, 3) = 369 Internal Sets



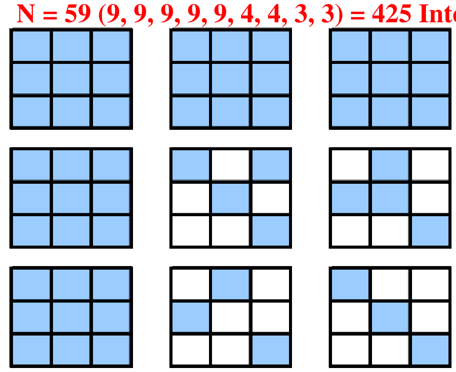
N = 57 (9, 9, 9, 9, 9, 9, 3, 3, 3, 3) = 388 Internal Sets



N = 58 (9, 9, 9, 9, 9, 4, 3, 3, 3) = 406 Internal Sets

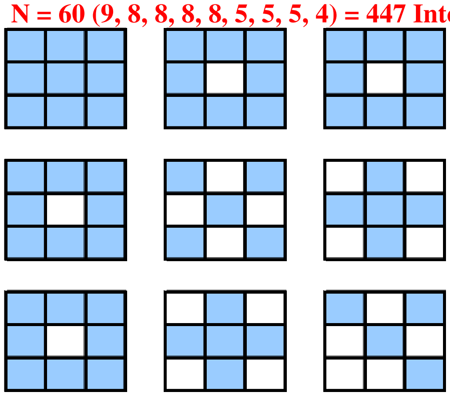


N = 59 (9, 9, 9, 9, 9, 4, 4, 3, 3) = 425 Internal Sets

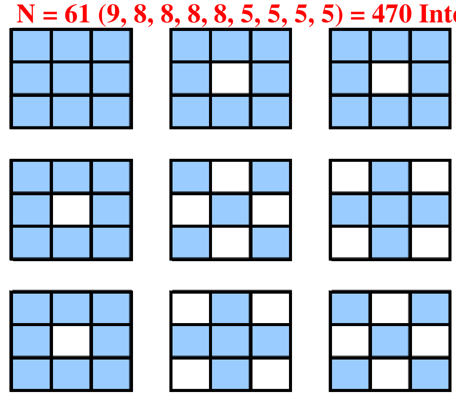


For N = 60, we make new choices to maximize the Internal Set count and use the 5-card block.

N = 60 (9, 8, 8, 8, 8, 5, 5, 5, 4) = 447 Internal Sets

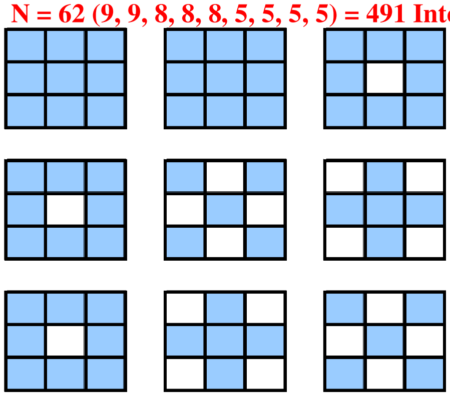


N = 61 (9, 8, 8, 8, 8, 5, 5, 5, 5) = 470 Internal Sets

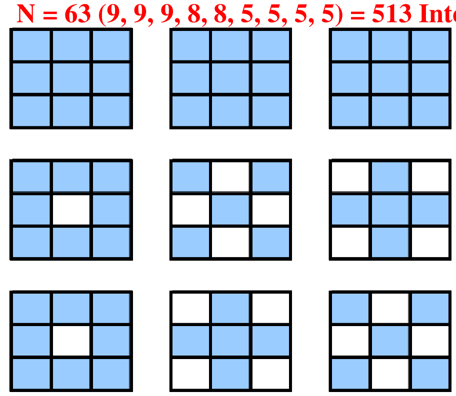


The choice for N = 62 and all remaining positions do not matter.

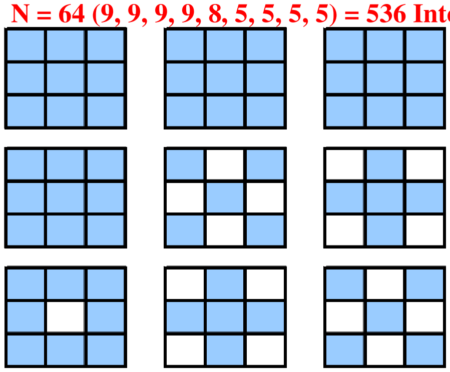
N = 62 (9, 9, 8, 8, 8, 5, 5, 5, 5) = 491 Internal Sets



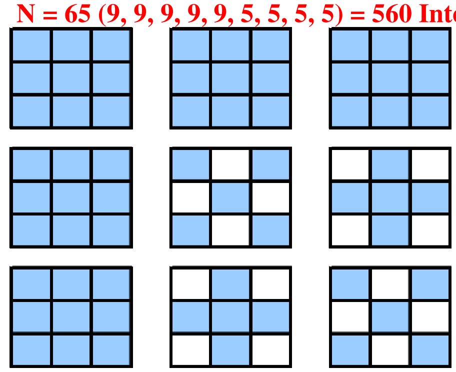
N = 63 (9, 9, 9, 8, 8, 5, 5, 5, 5) = 513 Internal Sets



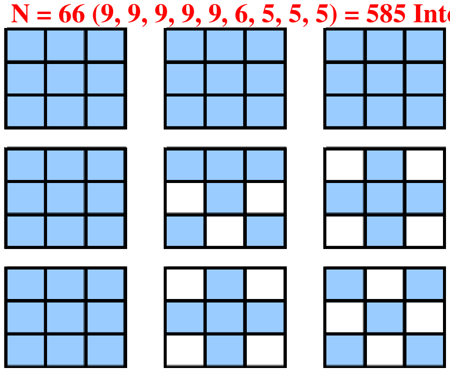
N = 64 (9, 9, 9, 9, 8, 5, 5, 5, 5) = 536 Internal Sets



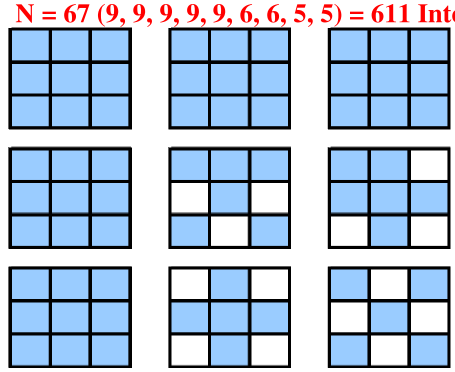
N = 65 (9, 9, 9, 9, 9, 5, 5, 5, 5) = 560 Internal Sets



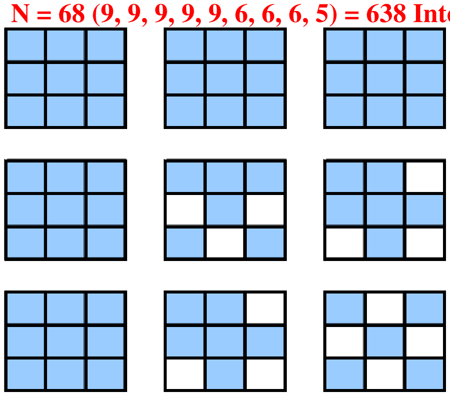
N = 66 (9, 9, 9, 9, 9, 6, 5, 5, 5) = 585 Internal Sets



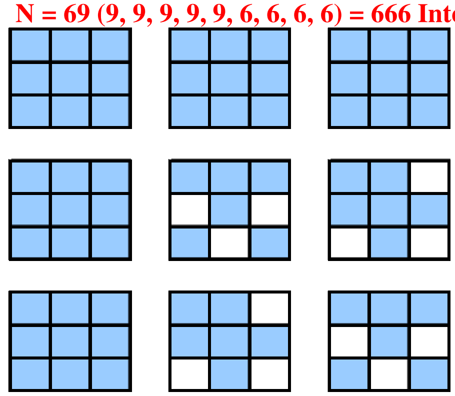
N = 67 (9, 9, 9, 9, 9, 6, 6, 5, 5) = 611 Internal Sets



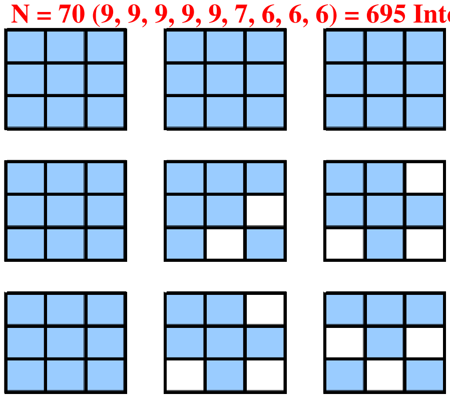
N = 68 (9, 9, 9, 9, 9, 6, 6, 6, 5) = 638 Internal Sets



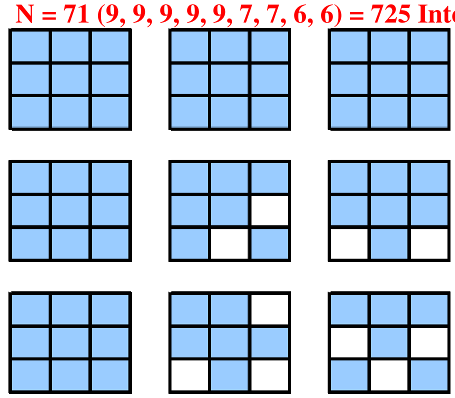
N = 69 (9, 9, 9, 9, 9, 6, 6, 6, 6) = 666 Internal Sets



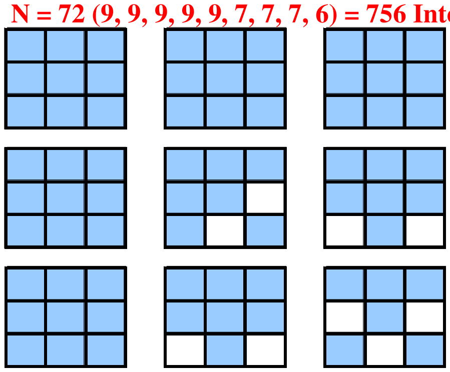
N = 70 (9, 9, 9, 9, 9, 7, 6, 6, 6) = 695 Internal Sets



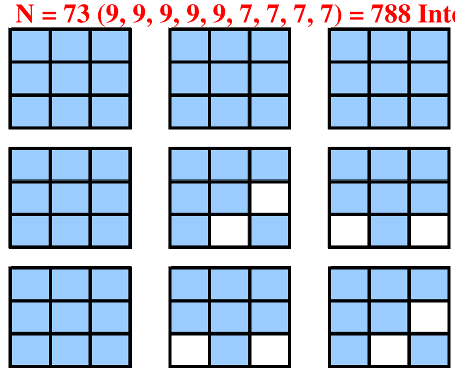
N = 71 (9, 9, 9, 9, 9, 7, 7, 6, 6) = 725 Internal Sets



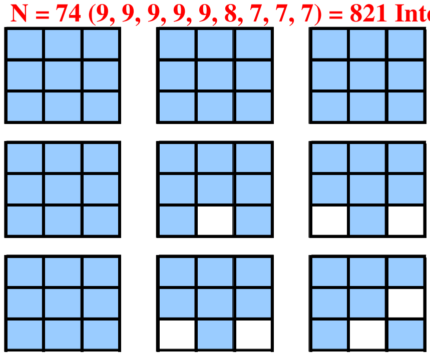
N = 72 (9, 9, 9, 9, 9, 7, 7, 7, 6) = 756 Internal Sets



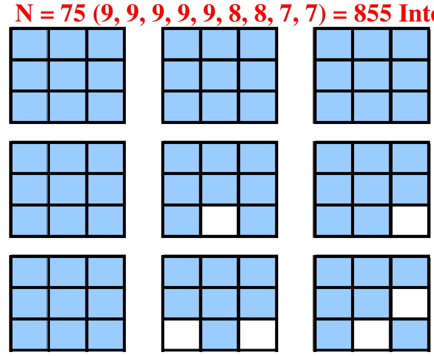
N = 73 (9, 9, 9, 9, 9, 7, 7, 7, 7) = 788 Internal Sets



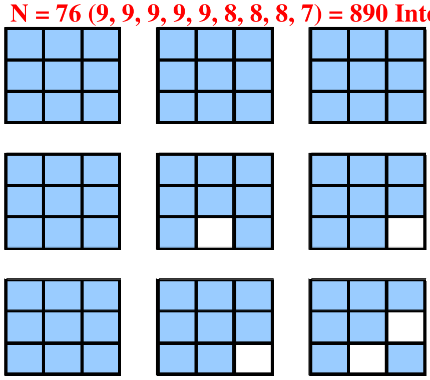
N = 74 (9, 9, 9, 9, 9, 8, 7, 7, 7) = 821 Internal Sets



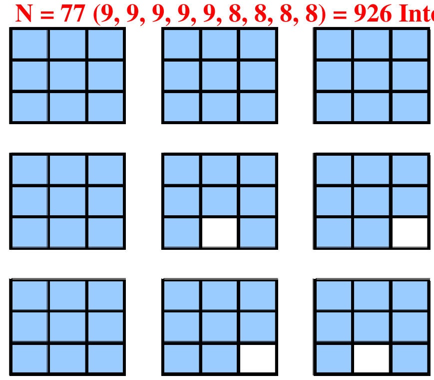
N = 75 (9, 9, 9, 9, 9, 8, 8, 7, 7) = 855 Internal Sets



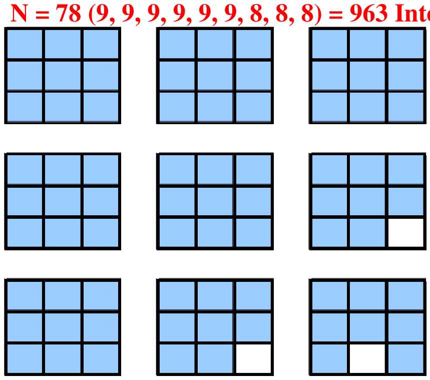
N = 76 (9, 9, 9, 9, 9, 8, 8, 8, 7) = 890 Internal Sets



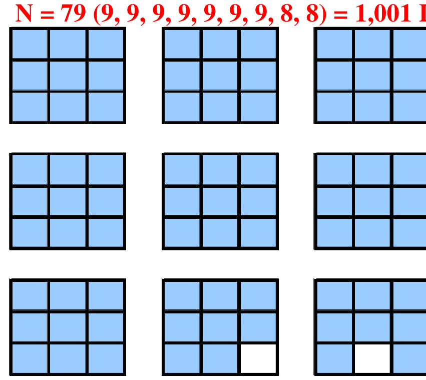
N = 77 (9, 9, 9, 9, 9, 8, 8, 8, 8) = 926 Internal Sets



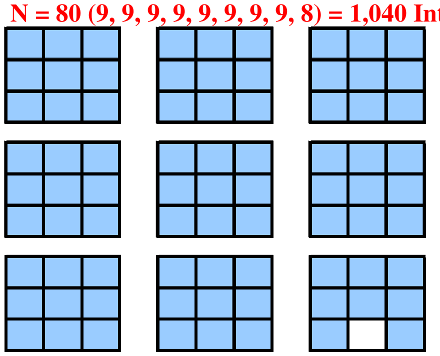
N = 78 (9, 9, 9, 9, 9, 9, 8, 8, 8) = 963 Internal Sets



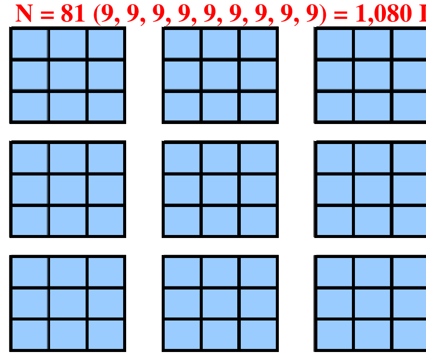
N = 79 (9, 9, 9, 9, 9, 9, 9, 8, 8) = 1,001 Internal Sets



N = 80 (9, 9, 9, 9, 9, 9, 9, 9, 8) = 1,040 Internal Sets



N = 81 (9, 9, 9, 9, 9, 9, 9, 9, 9) = 1,080 Internal Sets



APPENDIX II: KNOWN AND MODELED SOLUTIONS FOR THE MAXIMUM NUMBER OF SETLESS CARDS

Solution from Consecutive Maximization Method
P = 3 properties, N = 8 cards

1	4	6	2	7	5	3	8	9
10	12	14	17	20	22	16	21	23
18	24	26	11	13	15	19	25	27

Known Solution
P = 3 properties, N = 9 cards

X		X					X	
				X		X		X
X		X					X	

Solution from Consecutive Maximization Method
P = 4 properties, N = 16 cards

1	10	12	2	13	11	3	14	15
28	30	32	35	38	40	34	39	41
36	42	44	29	31	33	37	43	45
4	16	20	6	21	17	7	22	24
46	50	52	58	66	68	59	67	69
60	70	72	47	51	53	61	71	73
5	18	23	8	25	19	9	26	27
48	54	56	62	74	76	63	75	77
64	78	80	49	55	57	65	79	81

Known Solution
P = 4 properties, N = 20 cards

X		X					X	
				X		X		X
X		X					X	
	X						X	
	X					X		X
X		X		X				
	X					X		X

Solution from Consecutive Maximization Method
P = 5, N = 32 cards

1	28	30	2	31	29	3	32	33
82	84	86	89	92	94	88	93	95
90	96	98	83	85	87	91	97	99
4	34	38	6	39	35	7	40	42
100	104	106	112	120	122	113	121	123
114	124	126	101	105	107	115	125	127
5	36	41	8	43	37	9	44	45
102	108	110	116	128	130	117	129	131
118	132	134	103	109	111	119	133	135
10	46	50	12	51	47	13	52	54
136	140	142	148	156	158	149	157	159
150	160	162	137	141	143	151	161	163
16	58	66	20	67	59	21	68	70
172	180	182	196	212	214	197	213	215
198	216	218	173	181	183	199	217	219
17	60	69	22	71	61	23	72	74
174	184	186	200	220	222	201	221	223
202	224	226	175	185	187	203	225	227
11	48	53	14	55	49	15	56	57
138	144	146	152	164	166	153	165	167
154	168	170	139	145	147	155	169	171
18	62	73	24	75	63	25	76	78
176	188	190	204	228	230	205	229	231
206	232	234	177	189	191	207	233	235
19	64	77	26	79	65	27	80	81
178	192	194	208	236	238	209	237	239
210	240	242	179	193	195	211	241	243

Known Solution
P = 5, N = 45 cards

		X						
	X	X	X					
			X			X		X
X								
			X	X			X	
X								
X		X					X	
X						X		
			X					
	X		X		X		X	
						X		
			X					
	X				X			
					X			
				X		X		
		X						
					X			
	X	X			X			
	X		X	X				
	X							
		X			X		X	
	X	X		X	X			

Solution from Consecutive Maximization Method
P = 6, N = 64 cards

1	82	84	2	85	83	3	86	87	4	88	92	6	93	89	7	94	96	5	90	95	8	97	91	9	98	99
244	246	248	251	254	256	250	255	257	262	266	268	274	282	284	275	283	285	263	267	269	276	286	288	277	287	289
252	258	260	245	247	249	253	259	261	264	270	272	278	290	292	279	291	293	265	271	273	280	294	296	281	295	297
10	100	104	12	105	101	13	106	108	16	112	120	20	121	113	21	122	124	17	114	123	22	125	115	23	126	128
298	302	304	310	318	320	311	319	321	334	342	344	358	374	376	359	375	377	335	343	345	360	378	380	361	379	381
312	322	324	299	303	305	313	323	325	336	346	348	362	382	384	363	383	385	337	347	349	364	386	388	365	387	389
11	102	107	14	109	103	15	110	111	18	116	127	24	129	117	25	130	132	19	118	131	26	133	119	27	134	135
300	306	308	314	326	328	315	327	329	338	350	352	366	390	392	367	391	393	339	351	353	368	394	396	369	395	397
316	330	332	301	307	309	317	331	333	340	354	356	370	398	400	371	399	401	341	355	357	372	402	404	373	403	405
28	136	140	30	141	137	31	142	144	34	148	156	38	157	149	39	158	160	35	150	159	40	161	151	41	162	164
406	410	412	418	426	428	419	427	429	442	450	452	466	482	484	467	483	485	443	451	453	468	486	488	469	487	489
420	430	432	407	411	413	421	431	433	444	454	456	470	490	492	471	491	493	445	455	457	472	494	496	473	495	497
46	172	180	50	181	173	51	182	184	58	196	212	66	213	197	67	214	216	59	198	215	68	217	199	69	218	220
514	522	524	538	554	556	539	555	557	586	602	604	634	666	668	635	667	669	587	603	605	636	670	672	637	671	673
540	558	560	515	523	525	541	559	561	588	606	608	638	674	676	639	675	677	589	607	609	640	678	680	641	679	681
47	174	183	52	185	175	53	186	188	60	200	219	70	221	201	71	222	224	61	202	223	72	225	203	73	226	228
516	526	528	542	562	564	543	563	565	590	610	612	642	682	684	643	683	685	591	611	613	644	686	688	645	687	689
544	566	568	517	527	529	545	567	569	592	614	616	646	690	692	647	691	693	593	615	617	648	694	696	649	695	697
29	138	143	32	145	139	33	146	147	36	152	163	42	165	153	43	166	168	37	154	167	44	169	155	45	170	171
408	414	416	422	434	436	423	435	437	446	458	460	474	498	500	475	499	501	447	459	461	476	502	504	477	503	505
424	438	440	409	415	417	425	439	441	448	462	464	478	506	508	479	507	509	449	463	465	480	510	512	481	511	513
48	176	187	54	189	177	55	190	191	62	204	227	74	229	205	75	230	232	63	206	231	76	233	207	77	234	236
518	530	532	546	570	572	547	571	573	594	618	620	650	698	700	651	699	701	595	619	621	652	702	704	653	703	705
548	574	576	519	531	533	549	575	577	596	622	624	654	706	708	655	707	709	597	623	625	656	710	712	657	711	713
49	178	192	56	193	179	57	194	195	64	208	235	78	237	209	79	238	240	65	210	239	80	241	211	81	242	243
520	534	536	550	578	580	551	579	581	598	626	628	658	714	716	659	715	717	599	627	629	660	718	720	661	719	721
552	582	584	521	535	537	553	583	585	600	630	632	662	722	724	663	723	725	601	631	633	664	726	728	665	727	729